Problem. 1 Prove or Disprove the following:
If f is an entire function such that $|f(|z|^{1/3})| \leq 2+3|z|^{300}$ then f is a polynomial.

Proof. Counterexample:
We know that sin(z) is an entire function which is not a polynomial.
and $|\sin(|z|^{1/3})| \leq 1$ because $|z|^{1/3}$ is real. But the RHS of the inequality is always strictly greater than 1. Thus $|\sin(|z|^{1/3})| \leq 2+3|z|^{300}$ but it is not a polynomial. □

Problem. 2 Prove(in detail) that $\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\phi(t)dt$ is a holomorphic function of z on U for any continuous function ϕ on $[-\pi,\pi]$ with $\phi(-\pi)=\phi(\pi)$.

Problem. 3 Find the number of zeroes of the polynomial $1-2z^{10}+(3/4)z^n$ in U for any integer $n>10$.

Proof. Since the given function is a polynomial it is an entire function. So it is a analytic function on the unit circle.

$1-2z^{10}+(3/4)z^n$ has no zeroes on the unit circle as then we will have $1+(3/4)z^n = 2z^{10}$ but lhs has maximum absolute value to be 7/4 which is strictly less than the absolute value of the rhs which is 2.

Since $n>10$ we have $|z^n| < |z^{10}|$ in and thus $|1+(3/4)z^n| < 1+(3/4)z^{10} < |-2z^{10}| = 2|z^{10}|$ Hence by Rouche's theorem we have $1-2z^{10}+(3/4)z^n$ and $-2z^{10}$ have the same number of zeroes in U which is 10. □

Problem. 4 Show that $\Pi_{n=1}^{\infty}\frac{(1-\frac{1}{n^2})-z}{1-(1-\frac{1}{n^2})z}$ converges to a holomorphic function uniformly in compact subsets of U and the set of zeroes has a limit point at 1.

Proof. We use the following Theorem 15.6 from Real and Complex Analysis , Walter Rudin, 3rd edition, which says that:

Suppose $f_n \in H(\Omega)$ for $n=1,2\ldots$ such that no f_n is identically zero in any component of Ω, and $\sum_{n=1}^{\infty}|1-f_n(z)|$ converges uniformly on compact subsets of Ω.Then the product $f(z) = \Pi_{n=1}^{\infty} f_n(z)$ converges uniformly on compact subsets of Ω. Hence $f \in H(\Omega)$.Furthermore, we have $m(f;z) = \sum_{n=1}^{\infty} m(f_n;z)(z\in\Omega)$ where $m(f;z)$is defined to be the multiplicity of the zero of f at z.

Let $a_n = (1-\frac{1}{n^2})$.
So, here we have $1-\frac{a_n-z}{1-a_nz} = (1-\frac{a_n-z}{1-a_nz})\frac{a_n}{a_n} = \frac{(a_n-a_n^2z-a_n^2+a_nz)}{(1-a_nz)a_n} = \frac{(a_n+a_nz)(1-a_n)}{(1-a_nz)a_n}$
If $|z|\leq r$ for some $0<r<1$ then
$\frac{(a_n+a_nz)(1-a_n)}{(1-a_nz)a_n} \leq \frac{1+r}{1-r}(1-a_n)$
Hence we get $\sum_{n=1}^{\infty}|1-f_n(z)| \leq \sum_{n=1}^{\infty}\frac{1+r}{1-r}(1-a_n) = \frac{1+r}{1-r}\sum_{n=1}^{\infty}(\frac{1}{n^2}) < \infty$.
So we get that $\sum_{n=1}^{\infty}|1-f_n(z)|$ converges uniformly on compact subsets of U and hence $f(z) = \Pi_{n=1}^{\infty}\frac{(1-\frac{1}{n^2})-z}{1-(1-\frac{1}{n^2})z} \in H(U)$.

And, since the zeroes of f_n are $\{a_n\}$, the zeroes of f are also $\{a_n\}$ which has limit point 1. ◻

Problem. 5 Let f be a holomorphic function on $\mathbb{C} \setminus \{0\}$ such that f has a pole at 0 and $z^2 f(z)$ is bounded on $\{z : |z| \geq a\}$ for some positive number a. Prove that the residue of f at 0 is necessarily 0.

Proof. Let f has a pole at 0 of order $m > 0$.
Then we can write $f(z) = \frac{h(z)}{z^m}$ where $h(z)$ is analytic in $\mathbb{C}$ and $h(0) \neq 0$
Since, $z^2 f(z)$ is bounded on $\{z : |z| \geq a\}$ for some positive number a, we get that $\frac{f(\frac{1}{z})}{z^2}$ is bounded in $\{z : |z| < 1/a\}$.

Now, $g(z) = \frac{f(\frac{1}{z})}{z^2} = h(\frac{1}{z}) z^{m-2}$
Since $g(z)$ is bounded in $\{z : |z| < 1/a\}$ it can only have a removable singularity at 0.
But, $h(z) = \sum_{n=1}^{\infty} a_n z^n$ which says $h(\frac{1}{z}) = \sum_{n=1}^{\infty} a_n \frac{1}{z^n}$

if $g(z)$ do not have any singularity at 0 then a_n should be 0 $\forall n > m-2$
which forces $h(z) = \sum_{n=1}^{m-2} a_n z^n$ and thus $f(z) = \sum_{n=1}^{m-2} a_n z^{n-m}$ which shows that
$Res_0 f = a_{-1} = 0$

Now if $g(z)$ do have a removable singularity at 0 then $\lim_{z \to 0} z g(z) = 0$ which implies $\lim_{z \to 0} h(\frac{1}{z}) z^{m-1} = 0$ which again shows that a_n should be 0 $\forall n > m-2$
and the conclusion follows as above.

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Problem. 6 Find an entire function whose real part is $1 + 2x^2 - 2y^2 + 3x^3 - 9xy^2$

Proof. Let $u(x, y) = 1 + 2x^2 - 2y^2 + 3x^3 - 9xy^2$
. Then, $u_x = 4x + 9x^2 - 9y^2$ and $u_y = -4y - 18xy$
and hence, $u_{xx} = 4 + 18x$ and $u_{yy} = -4 - 18x$
Thus we can see that $u_{xx} + u_{yy} = 0$ which says that u is a harmonic function.
Then from the following theorem:

Let G be either the whole plane $\mathbb{C}$ or some open unit disk. If $u : G \to \mathbb{R}$ is a harmonic function then u has a harmonic conjugate.

We get a harmonic conjugate for the given $u(x, y)$ by the formula:
$v(x, y) = \int_0^y u_x(x, t) dt - \int_0^x u_y(s, 0) ds$

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Problem. 7 Find all holomorphic functions on $\mathbb{C} \setminus \{1\}$ such that f has a pole of order 3 at 1 and $\text{Re}[(z-1)^3 f(z)] \geq 3$ for all z.

Proof. Since f has a pole of order 3 at 1 then $f(z) = \frac{h(z)}{(z-1)^3}$, where $h(z)$ is analytic in $\mathbb{C}$ and $h(1) \neq 0$. Now its given that $\text{Re}[(z-1)^3 f(z)] \geq 3$ which implies $\text{Re}[h(z)] \geq 3$. But $h(z)$ is an entire function whose real part is bounded below, hence it is a constant.Thus the only possibilities of $f(z)$ is scalar multiples of $\frac{1}{(z-1)^3}$.

□

Problem. 8 Evaluate $\int_0^\infty \frac{x\sin(x)}{x^4+1}$ by the method of residues.

Proof. $\int_0^\infty \frac{x\sin(x)}{x^4+1} = Im(\int_0^\infty \frac{xe^{ix}}{x^4+1})$

Poles of the given function are $e^{i\theta}$ where $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
Let $a_n = \exp(i[\frac{\pi}{4} + (n-1)\frac{\pi}{2}]$ for $n = 1, 2, 3, 4$.

Now to find the given integral let us consider the contour γ_R be the path along the positive real axis starting from the origin upto some $R > (1/2)$ then circling back counter-clockwise through the upper half-plane to the origin, letting the circle get infinitely big.
So a_1 is the only pole inside γ_R
Thus,
$Im(\int_0^\infty \frac{xe^{ix}}{x^4+1}) = Im(2\pi i Res(f; a_1))$
where $f(x) = \frac{xe^{ix}}{x^4+1}$ Since a_1 is a simple poles of $f(x)$.
$Res(f; a_1) = \lim_{x \to a_1}(x - a_1)f(x) = a_1 e^{ia_1}(a_1 - a_2)^{-1}(a_1 - a_3)^{-1}(a_1 - a_4)^{-1} = \frac{1}{4}e^{ia_1}(a_1)^{-2}$
Hence, $Im(2\pi i Res(f; a_1)) = Im(2\pi i \frac{1}{4}e^{ia_1}(a_1)^{-2}) = Im(2\pi \frac{1}{4}e^{ia_1}) = \frac{\pi}{2e^{\frac{1}{\sqrt{2}}}}\sin(\frac{1}{\sqrt{2}})$.
Thus, $\int_0^\infty \frac{xsin(x)}{x^4+1} = \frac{\pi}{2e^{\frac{1}{\sqrt{2}}}}\sin(\frac{1}{\sqrt{2}})$.

□