1. Does there exist a non-constant entire function f such that $|f(z^3)| \leq 1 + |z|$ for all z?

Answer: Putting $z^3 = w$, rewrite the inequality as

$$|f(w)| \leq 1 + |w|^{\frac{1}{3}}$$

for all $w \in \mathbb{C}$. Here f is entire. So it has a power series expansion around zero. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be the power series expansion.

On a disc of radius R we have

$$|f(w)| \leq 1 + |w|^{\frac{1}{3}} \leq 1 + R^{\frac{1}{3}}.$$

Thus by Cauchy's estimate

$$\frac{|f^k(0)|}{k!} = |a_k| \leq \frac{1 + R^{\frac{1}{3}}}{R^k}.$$

This is true for any R (as f is entire) and hence $a_k = 0$ for any $k \geq 1$. Therefore f is constant. Hence there does not exist any non-constant entire function such that $|f(z^3)| \leq 1 + |z|$ for all z.

2. Prove that if $\gamma : [0, 1] \to \mathbb{C}$ is a continuously differentiable then $f(z) = \int_\gamma \frac{g(\zeta)}{\zeta - z} d\zeta$ defines a holomorphic function on $\mathbb{C} \setminus \gamma^*$ for any continuous function g on γ^*.

Answer: Since γ^* is compact and g is continuous and hence g is bounded on γ^*. It is easy to show that the function $f(z) = \int_\gamma \frac{g(\zeta)}{\zeta - z} d\zeta$ is continuous. Let $z \in \mathbb{C} \setminus \gamma^*$ and choose h such that $z + h \in \mathbb{C} \setminus \gamma^*$. Consider

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{1}{h} \int_\gamma g(\zeta) \frac{h}{(\zeta + h - z)(\zeta - z)} d\zeta \\ &= \int_\gamma g(\zeta) \frac{1}{(\zeta + h - z)(\zeta - z)} d\zeta \qquad (1) \end{aligned}$$

Taking $h \to 0$ and using the continuity we have from (1) that $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists and $f'(z) = \int_\gamma g(\zeta) \frac{1}{(\zeta - z)^2} d\zeta$. This completes the proof.

3. If $f \in C(\bar{U}) \cap H(U)$ and $|f(z) - 1 - 2z| < 1$ for $|z| = 1$, then prove that f has a unique zero in the unit disc U.

Answer: Let $g(z) = 2z + 1$. Then for $|z| = 1$, $|g(z)| \geq 1$. Since f is continuous on $\bar{U}$, we can extend f to be a holomorphic on an open region containing $\bar{U}$ and g is also holomorphic on that region. Therefore we have

$$|f(z) - g(z)| < |g(z)|$$

for all $|z| = 1$. Now g has only one zero in the disc. Using Rouche's theorem we conclude that f has only one zero in the disc U.

4. Let $z_n \in \mathbb{C} \setminus \{0\}$ for all n. Prove that $\prod_{n=1}^{\infty} z_n$ converges to a nonzero number if and only if $\sum_{n=1}^{\infty} Log(z_n)$ converges.

Answer: Suppose $\sum_{n=1}^{\infty} Log(z_n)$ converges. Let $s_n = \sum_{k=1}^{n} Log(z_k)$ and s_n converges to s. Then $exp(s_n) \to exp(s)$. But $exp(s_n) = \prod_{k=1}^{n} z_k$. Therefore $\prod_{n=1}^{\infty} z_n$ converges and converges to $exp(s) \neq 0$.

Conversely, suppose $\prod_{n=1}^{\infty} z_n$ converges to a nonzero number $z = re^{i\theta}$, $-\pi < \theta \leq \pi$. Then $z_n \to 1$ as $n \to \infty$. WLOG we can choose $Re(z_n) > 0$ for all n. Let $r_n = \prod_{k=1}^{n} z_k$ and $\ell(r_n) = Log|r_n| + i\theta_n$, where $\theta - \pi < \theta_n \leq \theta + \pi$. Now $exp(s_n) = r_n$ and hence $s_n = \ell(r_n) + 2\pi i k_n$ for some integer k_n. Again $s_n - s_{n-1} = Log(z_n) \to 0$ as $n \to \infty$. Also $\ell(p_n) - \ell(p_{n-1}) \to 0$. Hence $k_n - k_{n-1} \to 0$, but k_n are integers so there exists N and k such that $k_m = k_n = k$ for $m, n \geq N$. Hence $s_n \to \ell(z) + 2\pi i k$. Thus $\sum_{n=1}^{\infty} Log(z_n)$ converges.

5. Let f and g be entire functions, $\epsilon, \Delta \in (0, \infty)$ and $1 \leq |f(z)| \leq |g(z)||z|^{-1-\epsilon}$ for $|z| \geq \Delta$. Prove that the sum of the residues of $\frac{f}{g}$ at all its poles is 0.

Answer: We can rewrite the inequality as follows:

$$0 < \Delta^{1+\epsilon} \leq |z|^{1+\epsilon} \leq |z|^{1+\epsilon}|f(z)| \leq |g(z)|$$

for $|z| \geq \Delta$. This shows that $\frac{f}{g}$ does not have pole for $|z| \geq \Delta$. So it is enough to consider on $|z| < \Delta$. Now

$$\frac{1}{2\pi i} \int_{\partial\Delta} \frac{f}{g} = \sum_{k=1}^{N} Res(z_k, \frac{f}{g}),$$

where $\partial\Delta$ is the boundary of the region of radius Δ and z_k are the poles of $\frac{f}{g}$.

$$\frac{1}{2\pi} \int_{\partial\Delta} \frac{|f(z)|}{|g(z)|} dz \leq \frac{1}{2\pi} \int_{\partial\Delta} \frac{1}{|z|^{1+\epsilon}} dz \leq \frac{1}{\Delta^{\epsilon}}.$$

This is true for any $\Delta > 0$. Therefore we have the required result.

6. Let $\Omega = \{z : Re(z) > 0\}$. Give an example of a bijection from Ω onto U which is bi-holomorphic.

Answer: Consider a function from $\Omega \to U$ by $z \mapsto \frac{z-1}{z+1}$. Then $\left|\frac{z-1}{z+1}\right| = \frac{|z|^2 - 2Re(z)+1}{|z|^2+2Re(z)+1} < 1$ as $Re(z) > 0$. Clearly this map is bijective and holomorphic.

The inverse map from U to Ω is defined by $w \mapsto \frac{1+w}{1-w}$. For $|w| < 1$, $Re\left(\frac{1+w}{1-w}\right) = \frac{1-|w|^2}{1+|w|^2} > 0$. Clearly, it is holomorphic. This is the required example.

7. Evaluate $\int_{\gamma} \frac{3z^3+2}{(z-1)(z^2+9)} dz$, where γ is a circle of radius 4 with center 0.

Answer: Note that there are only three simple poles namely $1, 3i$ and $-3i$. From the Residue formula, we have

$$\int_{\gamma} f = 2\pi i[Res(f, 1) + Res(f, 3i) + Res(f, -3i)],$$

where $f(z) = \frac{3z^3+2}{(z-1)(z^2+9)}$. Now by simple calculation, we have

$$Res(f, 1) = \lim_{z \to 1} (z-1)f(z) = \frac{1}{2}$$
$$Res(f, 3i) = \lim_{z \to 3i} (z-3i)f(z) = \frac{-81i+2}{-18-6i}$$
$$Res(f, -3i) = \lim_{z \to -3i} (z+3i)f(z) = \frac{81i+2}{-18+6i}.$$

Therefore $\int_{\gamma} f = 6\pi i$.

8. Evaluate $\int_0^{\infty} \frac{x^2}{x^6+1} dx$ by the method of residues.

Answer. Since the integrand is an even function, $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx = 2\int_0^{\infty} \frac{x^2}{x^6+1} dx$. Let $f(z) = \frac{z^2}{z^6+1}$. Clearly f has simple poles at $z_k = exp(\frac{(2k+1)}{6})$ for $k = 0, 1, 2, 3, 4, 5$. Consider closed semicircle of radius $R > 1$

with center zero and traversed in anti clockwise. Then z_1, z_2 and z_3 are the poles inside the semicircle. Hence from the Residue formula, we have

$$\int_\gamma f = 2\pi i[Res(f, z_1) + Res(f, z_2) + Res(f, z_3)] = \frac{\pi}{3}.$$

Now applying the definition of the line integral,

$$\int_\gamma f = \int_{-R}^{R} \frac{x^2}{x^6+1} dx + \int_0^\pi \frac{R^2 e^{i2\pi t} Re^{it}}{1+R^6 e^{6it}} dt. \tag{2}$$

For $0 \leq t \leq \pi$, $1+R^6e^{6it}$ lies on the circle center at 1 of radius R^6. Hence $|1+R^6e^{6it}| \geq R^6 - 1$. Therefore

$$\left| \int_0^\pi \frac{R^2 e^{i2\pi t} iRe^{it}}{1+R^6 e^{6it}} dt \right| \leq \frac{\pi R^3}{R^6-1}$$

which tends to zero as $R \to \infty$. Therefore as $R \to \infty$, we have from (2)

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx = \int_\gamma f = \frac{\pi}{3}.$$

Hence

$$\int_0^\infty \frac{x^2}{x^6+1} dx = \frac{1}{2} \int_\gamma f = \frac{\pi}{6}.$$