**Midterm Solution**

***1. Let $f \in Hol(\mathcal{D})$, and let $L$ be a line in $\mathbb{C}$. If $f(z) \in L$ for all $z \in \mathcal{D}$, then prove that $f$ is constant on $\mathcal{D}$.***

***Answer: If $f$ is non constant holomorphic then $f(\mathcal{D}) = L$ should be open as a non constant holomorphic function is an open map. But line $L$ is not open in $\mathbb{C}$. Hence $f$ must be constant.***

***2. Evaluate $\int_{C_1(0)} \frac{cosz}{z} dz$ and $\int_{C_1(0)} \frac{sinz}{z^2-5} dz$.***

***Answer: Let $f(z) = cosz$. Clearly, $f$ is holomorphic on $\mathbb{C}$. Let $U$ be an open region containing $C_1(0)$ and its interior and $U \subset B_{\sqrt{5}}(0)$.***
***$(i)$ By Cauchy's integral formula, we have $\int_{C_1(0)} \frac{f(z)}{z} dz = f(0) = 1$.***
***$(ii)$ Since $\frac{sinz}{z^2-5}$ is holomorphic on $U$, therefore $\int_{C_1(0)} \frac{sinz}{z^2-5} dz = 0$.***

***4. Let $f \in Hol(\mathbb{C})$, and let $f(z_1 + z_2) = f(z_1) + f(z_2)$ for all $z_1, z_2$. Prove that there exists a scalar $\alpha$ such that $f(z) = \alpha z$ for all $z$.***

***Ans: By the given hypothesis we have for $n \in \mathbb{N}$***

$$f(n) = nf(1)$$

***and***

$$f(-n) = -nf(1).$$

***Now clearly***

$$f(\frac{m}{n}) = \frac{m}{n} f(1) \quad \textbf{\textit{for all}} \ \frac{m}{n} \in \mathbb{Q}.$$

***Since $\mathbb{Q}$ is dense in $\mathbb{R}$, and $f$ is continuous***

$$f(x) = xf(1) \quad \textbf{\textit{for all}} \ x \in \mathbb{R}.$$

***Take $g(z) = f(1)z$. Now consider***

$$Z = \{z \in \mathbb{C} : f(z) = g(z)\}.$$

***Then clearly, $Z$ contains $\mathbb{R}$ as well as limit point. Hence by identity theorem $f(z) = g(z)$. This finishes the proof.***

***5. Let $f \in Hol(\mathcal{D})$, and let $f$ has distinct zeros $z_1, ..., z_n$ with multiplicities $m_1, ..., m_n$, respectively. Prove that there exists $g \in Hol(\mathcal{D})$ such that***

$$f(z) = (z - z_1)^{m_1} \dots (z - z_2)^{m_n} g$$

***Ans: We can take power series expansion $f$ in $B_{r_1}\bar{(}z_1) \subset \mathcal{D}$, for some $r_1 > 0$. Since $f$ has zero at $z_1$, we have***

$$f(z) = (z - z_1)^{m_1} g_1(z)$$

***Now $g_1$ is also a power series and $f(z_2) = 0$ but $(z_2 - z_1) \neq 0$. So $g_1(z_2) = 0$ and $g_1(z_1) \neq 0$. Therefore***

$$g_1(z) = (z - z_1)^{m_2} g_2(z).$$

***Continuing in this way we have***

$$f(z) = (z - z_1)^{m_1} \cdots (z - z_n)^{m_n} g(z)$$

***for some holomorphic function*** $g$.

***6. Let*** $\mathcal{D}$ ***be a domain in*** $\mathbb{R}^2$ ***and let*** $u$ ***and*** $v$ ***be harmonic functions on*** $\mathcal{D}$. ***True or false (with justification): (i)*** $u+v$ ***is harmonic. (ii)*** $uv$ ***is harmonic.***

***Ans: Let*** $u$ ***and*** $v$ ***be harmonic functions on*** $\mathcal{D}$. ***Then***

$$u_{xx} + u_{yy} = 0$$

***and***

$$v_{xx} + v_{yy} = 0,$$

***where*** $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ ***and*** $u_{yy} = \frac{\partial^2 u}{\partial y^2}$. ***Clearly from the above equation we have***

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u+v) = 0.$$

***This proves that*** $u+v$ ***is harmonic.***

$(ii)$ ***Consider*** $u(x,y) = v(x,y) = xy$. ***Then***

$$u_{xx} + u_{yy} = 0.$$

***Therefore*** $u$ ***is harmonic. Similarly*** $v$ ***is also harmonic. But***

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(uv) = 2(y^2 + x^2) \neq 0.$$

***Hence product of two harmonic functions need not be harmonic.***

***7. Prove that for all polynomial*** $p \in \mathbb{C}[z]$,

$$\sup_{z \in C_1(0)} |z^{-1} - p(z)| \geq 1.$$

***Ans: Note that***

$$\sup_{z \in C_1(0)} |z^{-1} - p(z)| = \sup_{z \in C_1(0)} |1 - zp(z)|.$$

***Let*** $f(z) = 1 - zp(z)$ ***for*** $z \in B_1(0) = \{z \in \mathbb{C} : |z| < 1\}$. ***Then*** $f$ ***is holomorphic on*** $B_1(0)$. ***Also*** $f(0) = 1$. ***So by maximum modulus principle*** $\sup_{z \in C_1(0)} |f(z)| \geq 1$. ***This completes the proof.***

***8. Let*** $f$ ***be a non-constant entire function. (i) Prove that the range of*** $f$ ***is dense. (ii) If*** $|f| = 1$ ***on*** $C_1(0)$, ***then describe*** $f$.

***Ans: (i) Suppose range of*** $f$ ***is not dense. Then there exists*** $w \in \mathbb{C}$ ***and*** $r > 0$ ***such that*** $|f(z) - w| > r$ ***for every*** $z \in \mathbb{C}$. ***Consider***

$$h(z) = \frac{1}{f(z) - w} \quad z \in \mathbb{C}.$$

***Clearly,*** $h$ ***is entire and***

$$|h(z)| = \frac{1}{|f(z) - w|} < \frac{1}{r}.$$

***Hence by Liouville's theorem*** $h(z)$ ***is constant. Therefore*** $f(z)$ ***is constant on*** $\mathbb{C}$

***9. Let*** $a_n \subseteq \mathbb{C}$ ***and let***

$$\sum_{n=0}^{\infty} |a_n| < \infty$$

***and let***

$$\sum_{n=0}^{\infty} \frac{a_n}{k^n} = 0$$

***for all integer*** $k \leq 2$***. Prove that*** $a_n = 0$ ***for all*** $n$.

***Answer: Suppose there exists a smallest integer*** $k$ ***such that*** $a_k \neq 0$***. Consider***

$$\sum_{n=k}^{\infty} a_n z^n = f(z).$$

***Since*** $\sum_{n=0}^{\infty} |a_n| < \infty$ ***so*** $a_n \to 0$ ***as*** $n \to \infty$***. Therefore*** $\limsup |a_n|^{\frac{1}{n}} \leq 1$***. Hence the radius of convergence of*** $f$ ***is bigger than equal to*** $1$***. Now*** $B_{\frac{1}{2}}(0)$ ***is inside the area of convergence for*** $f$***. Hence*** $f$ ***is holomorphic on*** $B_{\frac{1}{2}}(0)$***. Now consider***

$$Z(f) = \{z \in B_{\frac{1}{2}}(0) : f(z) = 0\}$$

***Clearly*** $Z(f)$ ***contains the set*** $\{\frac{1}{n} : n \geq 3\}$***. Therefore*** $Z(f)$ ***has limit point , namely 0 in*** $B_{\frac{1}{2}}(0)$***. Hence by Identity theorem*** $f = 0$***. This completes the proof.***