Complex Analysis Solutions *

Mid-Semester 2014-2015

Problem 1

(i) True.
The radius of convergence of the given function is $\left(\limsup_{n\to\infty}(\frac{2^n}{n!})^{\frac{1}{n}}\right)^{-1} = \infty$. The function $\sum_{n=0}^{\infty} \frac{2^n z^{3n}}{n!}$ is the power series expansion of e^{2z^3}, which is an entire function.

(ii) True.
Consider the contour formed by $C_e(0)$. Let $f(z) = 1 + ez + e^z$. Then from the Cauchy integral formula we have $f^{(2)}(1) = \frac{2!}{2\pi i}\int_{C_e(0)} \frac{f(z)}{(z-1)^3}dz$. From the choice of f, we have $f^{(2)}(1) = e^1 = e$. Therefore give assertion is true.

Problem 2

Choose $r > 0$, such that there is no zero of f in $B_r(z_0)$ other than z_0. Because f has a zero of order m at z_0, we can write $f(z) = (z - z_0)^m g(z)$. Therefore in $B_r(z_0)$, f can be written as $\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}$. By the choice of $B_r(z_0)$, $\frac{g'}{g}$ is analytic in $B_r(z_0)$ and hence $\int_{C_r(z_0)} \frac{g'(z)}{g(z)}dz = 0$. From the Cauchy integral formula we have $\int_{C_r(z_0)} \frac{m}{z-z_0} = 2\pi im$. Therefore for this choice of $C_r(z_0)$, we have shown that $\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z)}{f(z)} = m$.

Problem 3

The radius of convergence for the power series $\sum_{k=0}^{\infty} a_k z^k$ is $R = (\limsup_{n\to\infty} |a_n|^{\frac{1}{n}})^{-1}$. Given that $\sum_{k=0}^{\infty} |a_k|^2 < \infty$, therefore for any fixed $1 > \epsilon > 0$, we have N_ϵ such that $|a_n| < \epsilon$ for any $n > N_\epsilon$. Therefore $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq 1$ and hence $R \geq 1$. Therefore the given series is analytic in $B_1(0)$ and hence holomorphic.

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Problem 4

(i) Because $f(\mathbb{C}) \cap B_1(0)$ is empty, f doesn't have any zero, hence the function $\frac{1}{f}$ is well defined and is entire. From the hypothesis we get $|\frac{1}{f}| \leq 1$. We know that any bounded entire function is a constant function. Therefore f is a constant function, with the absolute value of the constant being at-least 1.

(ii) Consider the function g, defined as $g(z) = f(z) - f(z + 2\pi)$. Given that f is 2π periodic when restricted to real line. Therefore $g(z) = 0$, whenever $z \in \mathbb{R}$. But $\mathbb{R}$ is not a discrete set in $\mathbb{C}$, hence $g \equiv 0$. Therefore we get $f(z) = f(z+2\pi)$ for any $z \in \mathbb{C}$

Problem 5

The function $e^{f(z)}$ cannot assume the value 0. Therefore 0 is not in the domain U. Solving the equation $e^{f(z)} = z$, we get $f(z) = \log z$. Because $0 \notin U$ and f is continuous, for any $z \in U$, the function $f(z) = \log z$ (choose the principal branch of logarithm) is well defined in a small enough neighborhood of z, which also analytic in that neighborhood. Therefore f is holomorphic in U. Differentiating the equation $e^{f(z)} = z$ on both the sides, we get $f'(z)e^{f(z)} = 1$. Now substituting the given identity, we get $zf'(z) = 1$. Therefore $f'(z) = \frac{1}{z}$.

Problem 6

Let $I = \int\limits_\gamma \overline{f(z)} f'(z) dz = \int\limits_\gamma \overline{f(z)} df(z)$. Then, $\overline{I} = \int\limits_\gamma f(z) d\overline{f(z)}$.

$$2Re(I) = I + \overline{I} = \int\limits_\gamma (f(z)d\overline{f(z)} + \overline{f(z)}df(z)) = \int\limits_\gamma d(f(z)\overline{f(z)}) = 0.$$

Therefore I is purely imaginary.

Problem 7

Consider the following integral identity.

$$\int\limits_0^1 (z-w)e^{w+t(z-w)} dt = e^z - e^w.$$

Because $Re(z) < 0$ and $Re(w) < 0$, we have $|e^{w+t(z-w)}| \leq 1$. Therefore,

$$|e^z - e^w| = |\int\limits_0^1 (z-w)e^{w+t(z-w)} dt| \leq \int\limits_0^1 |(z-w)e^{w+t(z-w)}| dt \leq |z-w|.$$

Problem 8

Given $f \in Hol(\mathbb{C})$ and $f''(\frac{1}{n}) + f(\frac{1}{n}) = 0$ for all $n \geq 1$. Because $f \in Hol(\mathbb{C})$, we have $f'' + f \in Hol(\mathbb{C})$. By continuity of $f'' + f$, we have $f''(0) + f(0) = 0$. But, zeros of non-trivial holomorphic function are discrete and 0 is a limit point of the set $\{\frac{1}{n} : n \geq 1\}$. Therefore, we have $f'' + f \equiv 0$. Because f is an entire function, let the power series expansion of f be $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Substituting the power series expansion of f in the identity $f'' + f \equiv 0$ we get,

$$a_n + (n+2)(n+1)a_{n+2} = 0 \text{ for every } n \geq 0.$$

By solving these equation recursively we obtain

$$a_{2n} = (-1)^n \frac{a_0}{(2n)!} \text{ and } a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!} \text{ for every } n \geq 0.$$

Therefore $f(z) = a_0 cos(z) + a_1 sin(z)$. These are the only functions that satisfy the given property.