Complex Analysis Solutions *

Mid-Semester 2012-2013

Problem 1

From the Cauchy integral formula we know that,

$$\frac{d^3}{dz^3}f(1) = \frac{(-1)^3}{3!2\pi i}\int\limits_{|z-1|=r}\frac{f(z)}{(z-1)^4}dz.$$

For any $r > 2$, we have

$$\begin{aligned}\Big|\frac{d^3}{dz^3}f(1)\Big| \leq \frac{1}{12\pi}\int\limits_{|z-1|=r}\frac{|f(z)|}{|z-1|^4}|dz| &\leq \frac{1}{12\pi}\int\limits_{|z-1|=r}\frac{1+|z|^2}{|z-1|^4}|dz| \\ &\leq \frac{1}{12\pi}\int\limits_{|z-1|=r}\frac{2+|z-1|^2}{|z-1|^4}|dz| \\ &= \frac{1}{12\pi}\frac{(2+r^2)2\pi r}{r^4}\end{aligned}$$

Given any $\epsilon > 0$ by choosing r large, we get $\big|\frac{d^3}{dz^3}f(1)\big| < \epsilon$. Therefore $\frac{d^3}{dz^3}f(1) = 0$. Using similar calculation one can show $\frac{d^n}{dz^n}f(z) = 0$ for any $k \geq 3$ and $z \in \mathbb{C}$.

Problem 2

Given $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ converges whenever $|z| < 1$ and $f(\frac{1}{n}) \in \mathbb{R}$ for $n \geq 2$. To show that $f(\mathbb{R}) \subset \mathbb{R}$, it is enough to show that all a_n are real. Because f is continuous, we have

$$a_0 = \lim_{n\to\infty} f(\frac{1}{n}) = f(0) \in \mathbb{R}.$$

Define $f_k(z) = \frac{f_{k-1}(z)-f_{k-1}(0)}{z}$, for $k \geq 1$. Notice that $f_k(0) = a_k$ and $f_k(z) = \sum\limits_{n=0}^{\infty} a_{n+k}z^n$ for $k \geq 1$. We prove that $a_k \in \mathbb{R}$ for $k \geq 1$ by using induction. We have already shown that $a_0 \in \mathbb{R}$. If $a_{k-1} \in \mathbb{R}$, then from the definition $f_k(\frac{1}{n}) \in \mathbb{R}$ for every $n \geq 2$. Also notice that f_k is continuous in the set $\{z : |z| < 1\}$. Therefore $a_k = f_k(0) \in \mathbb{R}$.

*Send an email to tulasi.math@gmail.com for any clarifications or to report any errors.

Problem 3

The stereographic map evaluated at the point z is given by $x = \frac{2Re(z)}{|z|^2+1}, y = \frac{2Im(z)}{|z|^2+1}, z = \frac{|z|^2-1}{|z|^2+1}$. The points i and ∞ are mapped to $(0,1,0)$ and $(0,0,1)$ respectively. Therefore the stereographic distance $d(i,\infty) = \sqrt{0^2+1^2+1^2} = \sqrt{2}$. In general, the stereographic distance between two points z, w is given by

$$d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}.$$

Problem 4

Denote $B_r(a)$ be the ball of radius r with center at a. We will first show the mean value property for polynomials of the form $P(z) = z^n$ for $n \geq 1$. Recall the Jacobian for transforming the real and imaginary co-ordinates to polar co-ordinates $(x+iy \to re^{i\theta})$ is r. Therefore,

$$\frac{1}{\pi r^2}\int\limits_{B_r(0)} z^n dm(z) = \frac{1}{\pi r^2}\int\limits_0^r\int\limits_0^{2\pi} r^n e^{in\theta} d\theta r dr = \frac{1}{\pi r^2}\int\limits_0^r r^{n+1} dr \int\limits_0^{2\pi} e^{in\theta} d\theta = 0 = P(0).$$

Therefore if $P(z) = \sum\limits_{k=0}^{n} a_k z^k$, then

$$\begin{aligned}\frac{1}{\pi r^2}\int\limits_{B_r(0)} P(z)dm(z) &= \frac{1}{\pi r^2}\int\limits_{B_r(0)} \sum_{k=0}^{n} a_k z^k dm(z) \\ &= \frac{1}{\pi r^2}\int\limits_{B_r(0)} a_0 dm(z) + \sum_{k=1}^{n}\int\limits_{B_r(0)} z^k dm(z) \\ &= a_0 = P(0).\end{aligned}$$

We now have the mean value property for any polynomial at the point 0. Now to show the mean value property at any arbitrary point a, shift the origin to a and obtain a corresponding new polynomial. Because Lebesgue measure is translation invariant, the mean value of the original polynomial at the point a is same as the mean value of the new polynomial at 0. Hence the mean value property holds for every polynomial at every point.

Problem 5

Given that $f, g, \overline{f}g \in H(\Omega)$, where Ω is a connected open set. Denote $h = \overline{f}g$. If $g \not\equiv 0$, then the zero set of g is a discrete set. Hence there is a open set $V \subset \Omega$ where g doesn't vanish. On the set V, the function $\frac{h}{g}$ is well-defined. Because $h, g \in H(V)$, we have $\frac{h}{g} = \overline{f} \in H(V)$. Therefore $Re(f) = f + \overline{f} \in H(\Omega)$. From open mapping theorem we have that the image of any open set under a non-constant holomorphic mapping is also open. Any subset of real line is not open in complex plane. Hence the $Re(f)$ is constant on V. Similarly, we can show that $Im(f)$ is constant on V. Therefore, f is constant on V and is constant on Ω.

Problem 6

Let f is not a constant function. Let there is $z \in \mathbb{C}$, such that $f(z) \neq f(c)$. Let U be a neighborhood of c containing z. Then, $f(U)$ is not open, because any neighborhood of $f(c)$ contains a number whose absolute valued exceeds $|f(c)|$, but it was given that $|f(z)| \leq |f(c)|$ for any $z \in \mathbb{C}$. Therefore from the open mapping theorem it follows that f is a constant function.

Problem 7

Let $f_n(z) = \prod_{k=1}^{n}(1 + \frac{z}{k^2} - \frac{z^2}{k^3})$ and $f(z) = \prod_{n=1}^{\infty}(1 + \frac{z}{n^2} - \frac{z^2}{n^3})$. f_ns are all analytic functions. Fix any compact set $K \subset \mathbb{C}$, then K is bounded (say by M). For any $z \in K$ and $n \geq 1$, we get

$$\begin{aligned} |1 + \frac{z}{n^2} - \frac{z^2}{n^3}| &\leq |1 + \frac{|z|}{n^2} + \frac{|z|^2}{n^3}| \\ &\leq 1 + \frac{M}{n^2} + \frac{M^2}{n^3} \\ &\leq e^{\frac{M}{n^2} + \frac{M^2}{n^3}}. \end{aligned}$$

Therefore,

$$|\prod_{n=1}^{\infty}(1 + \frac{z}{n^2} - \frac{z^2}{n^3})| \leq e^{\sum_{n=1}^{\infty}(\frac{M}{n^2} + \frac{M^2}{n^3})} < \infty.$$

The functions are uniformly bounded in any compact set K. By similar arguments, it can be shown that f_n converge point wise. From these facts one can verify the hypothesis of Morera's theorem in the disk where $\{z : |z| < M\}$ The zeros of $1 + \frac{z}{n^2} - \frac{z^2}{n^3}$ are $\frac{n}{2}(1 \pm \sqrt{1 + 4n})$. Therefore the set of zeros of the given infinite product are $\{\frac{n}{2}(1 \pm \sqrt{1 + 4n}) : n \in \mathbb{N}\}$.

Problem 8

Given $f(z) = \frac{2z-i}{2+iz}$. f is a rational function and is holomorphic on the set where the denomiator is non-zero. Therefore f is holomorphic on $\mathbb{C} \backslash \{2i\}$.

$$|f(z)|^2 = |\frac{(2z - i)^2}{(z + 2i)^2}| = \frac{4|z|^2 + 1 + 4Re(iz)}{|z|^2 + 4 + 4Re(iz)}.$$

The numerator of the right hand side of 1 is smaller than (equal to) the denominator whenever $|z| < 1 (= 1)$. Therefore f maps U into U (T into T).
Letting $f(z) = w$, we get $z = \frac{2w+i}{2-iw}$. Therefore $f^{-1}(z) = \frac{2z+i}{2-iz}$. By similar computation as in 1, we get

$$|f^{-1}(z)|^2 = \frac{4|z|^2 + 1 - 4Re(iz)}{|z|^2 + 4 - 4Re(iz)}.$$

Therefore f^{-1} also maps U into U and T into T. Combining the results for f and f^{-1} we see that f maps U onto U and T onto T.

Remark: The above property holds for any map of the form $f(z) = \frac{z-a}{1-\bar{a}z}$, whenever $a \in U$.