Mid-Semestral Exam 2009-2010
Complex Analysis

August 25, 2016

Problem 1. Prove or disprove the following:
if f is an entire function and g(z) = $\overline{f(\bar{z})}$(where $\bar{a}$ is the complex conjugate of a)then g is also an entire function.

Proof. Suppose f(z)=u(x,y)+iv(x,y), then g(z)= $\widetilde{u}(x,y)$+i$\widetilde{v}(x,y)$. Where $\widetilde{u}(x,y)$=u(x,-y) and $\widetilde{v}(x,y)$ = -v(x,-y).
Now we shall use the Cauchy-Riemann Equations to show that g(z) is an analytic function.
$\widetilde{u}_x = \frac{\partial u}{\partial x}$(x,-y)=$\frac{\partial v}{\partial y}$(x,-y)=-$\frac{\partial v}{\partial y}$(x,-y)=$\widetilde{v}_y$
similarly, $\widetilde{u}_y = \frac{\partial u}{\partial y}$(x,-y)=$\frac{\partial v}{\partial x}$(x,-y)=-$\{-\frac{\partial v}{\partial x}$(x,-y)$\}$=$-\widetilde{v}_x$
Hence g(z) is an analytic function.

□

Problem 2. Find all entire functions f such that $[f(z)]^3$ = e^z for all $z \in \mathbb{C}$.

Proof. $[f(z)]^3$ = e^z implies that $f(z) \neq 0$ for all $z \in \mathbb{C}$
from this it follows that $\frac{3f'(z)f(z)^2}{[f(z)]^3}$=1 and hence we get $f'(z) = \frac{f(z)}{3}$.

Thus, $f^n(z) = \frac{f(z)}{3^n}$, so at z=0 we have $f^n(0) = \frac{f(0)}{3^n}$ = $\frac{1}{3^n}$.

Since f is given to be an entire function, f(z)= $\sum_{n=0}^{\infty} a_n z^n$ where $a_n = \frac{f^n(0)}{n!}$, so from the above we have $a_n = \frac{1}{3^n n!}$.
Thus, f(z) = $\sum_{n=0}^{\infty} \frac{(z/3)^n}{n!} = e^{\frac{z}{3}}$.

Thus all the entire functions f such that $[f(z)]^3$ = e^z are of the form $e^{\frac{z+2k\pi i}{3}}$ for some k∈ N∪$\{0\}$.

□

Problem 3. If f(z) = $\frac{z}{1+z}$ find f (U). Is f a conformal equivalence of U onto f(U)?
Hint: use properties of Mobius transformations.

Proof. The inverse of the given f(z) is $g(w) = \frac{w}{1-w}$. Now, suppose f(z)=w, then z=g(w), i.e. $z = \frac{w}{1-w}$. Since $z \in U, |z| < 1$ and hence $|w| < |1 - w|$.

Let w=x+iy. Then we get from the above inequality that $(x^2+y^2) < (1-x)^2+y^2$ which implies $x < 1/2$. We also notice that f(0)=0.

Now f(z) being a Moebius transformation will always take connected components to connected components and hence f(U) will be the region $\{z \in \mathbb{C} : Re(z) < 1/2\}$ i.e. the left half plane with respect to the line x=1/2 in $\mathbb{R}^2$

The fact that f is one-one and analytic on U is easy to check and hence f(U) is conformally equivalent to U via f. □

Problem 4. Let γ be a continuously differentiable map from [0,1] into $\mathbb{C}$ with $\gamma(0) = 1$ and $\gamma(1) = i$. Evaluate $\int_\gamma 23 - 3z^5 + 7z^6 + 200z^{100}\, dz.$

Proof. Let Γ be the closed curve formed by joining the given curve γ and the unit circular arc from i to 1.
i.e. $\Gamma = \gamma(2t), \forall t \in [0,1]; \widetilde{\gamma}(t), \forall t \in [1, \pi/2]$ where $\widetilde{\gamma}(t) = e^{i[\frac{\pi}{2} - \frac{(t-1)(\pi/2)}{\frac{\pi}{2}-1}]}$. Now since the given integral is a polynomial and hence an entire function over $\mathbb{C}$ we have $\int_\Gamma 23 - 3z^5 + 7z^6 + 200z^{100}\, dz = 0$
But, $\int_\gamma 23-3z^5+7z^6+200z^{100}\, dz.$ = $\int_\gamma 23-3z^5+7z^6+200z^{100}\, dz.$ + $\int_{\widetilde{\gamma}} 23-3z^5+7z^6+200z^{100}\, dz.$
Therefore, $\int_\gamma 23 - 3z^5 + 7z^6 + 200z^{100}\, dz.$ = - $\int_{\widetilde{\gamma}} 23 - 3z^5 + 7z^6 + 200z^{100}\, dz.$
Now we calculate $\int_{\widetilde{\gamma}} 23-3z^5+7z^6+200z^{100}\, dz.$ using the relation $\int_{\widetilde{\gamma}} f(z)\, dz = \int_{1/2}^{\pi/2} f(\widetilde{\gamma}(t))\widetilde{\gamma}'(t)\, dt.$

□

Problem 5. Prove that if p is a non-constant polynomial of degree n then $\{z : |p(z)| < 1\}$ is a bounded open set with atmost n connected components. Give an example to show that the number of components can be less than n.

Proof. The boundedness follows from the fact that $|p(z)| \to \infty$ as $|z| \to \infty$ and the second property follows from the fact that p(z) is continous as it is holomorphic and the given region say G is the inverse of an open set under $p(z)$.

The main idea lies in the fact that each component of G will atleast have a root of $p(z)$.

Suppose C is a component of G not containing a root of $p(z)$, then $p(z) \neq 0$ for all z $\in$ C and thus $\frac{1}{p(z)}$ is holomorphic in C. Now since $\partial C \subset \partial G$ and $\partial G = \{z : |p(z)| = 1\}$ [Proof: let z be such that$|p(z)| = 1$, then if there are no sequence $\{z_n\}$ in G which converges to z, then it will violate the Maximum Modulus principle] we have $\frac{1}{|p(z)|} < 1$ and $p(z) < 1$ by the Maximum Modulus principle, which is a contradiction.

Thus each component of G will atleast have a root of $p(z)$ and hence by the Fundamental Theorem of Algebra, G can have atmost n many connected components.

This idea also shows that if we take a polynomial with multiple roots, number of components of G will be less than n.

□

Problem 6. If f is an entire function such that $|f(z)| \geq |z|$ for all z, prove that f is necessarily a polynomial.

Proof. Since $|f(z)| \to \infty$ as $|z| \to \infty$, f(z) has a pole at infinity, say of order m.
Now,an entire function having a pole at infinity of order m is necessarily a polynomial of degree m.
This follows from the fact that f has a pole at infinity of order m if f(1/z) has a pole at zero of order m.
So, we get that f(1/z) = $\sum_{n=-m}^{\infty} a_n z^n$, therefore f(z) = $\sum_{n=-m}^{\infty} \frac{a_n}{z^n}$
But since f is given to be an entire function, in the above expression we have $a_n = 0$ for $n \geq 1$. Thus we get that f(z) is a polynomial. □

Problem 7. Let f ∈ H(Ω) and f(z) $\notin (-\infty; 0]$ for all z ∈ Ω.Prove that log $|f|$ is a harmonic function on Ω.Also prove that the conclusion is true for any f ∈ H(Ω) such that f(z) ∉ 0 for all z ∈ Ω .

Proof. In the slit plane $\Omega = \mathbb{C} - \{(-\infty; 0]\}$ we have the principal branch of logarithm , i.e. log $f(z)$=ln $|f(z)|$ + i $(arg(f(z)))$ with $|arg(f(z))| < \pi$. Thus ln $|f(z)|$ is a harmonic function since it is the real part of an analytic function log $f(z)$.

For the second part we use the following result from page 65 of Complex Analysis in One Variable by Raghavan.

Let $\Omega \subset \mathbb{C}$ be a simply connected open set. Suppose that f is nowhere zero on Ω. Then there exists g ∈H(Ω) such that $e^g = f$.
This actually tells that g which is the branch of logarithm of f(z) is also the primitive of $\frac{f'}{f}$
Since it is given that f(z) $\neq 0$ for all z $\in \Omega$ the primitive of $\frac{f'}{f}$ exists and which is nothing but log f(z) and hence log $f(z)$=ln $|f(z)|$ + i $(arg(f(z)))$ is holomorphic on Ω. Then again ln $|f(z)|$ is harmonic as it is the real part of an analytic function. □