1. Find a Möbius transformation S from $\mathbb{R} \cup \{\infty\}$ to $\{z : |z| = 1\}$ which is surjective. Find the image under this transformation of $\{z : Im(z) > 0\}$.
Answer: Consider a Möbius transformation F from $\{z \in \mathbb{C} : |z| = 1\}$ to $\mathbb{R} \cup \{\infty\}$ by

$$F(z) = \frac{az+b}{cz+d}$$

where $ad - bc \neq 0$. By this transformation the points $i \to \infty$, $-i \to 0$ and $1 \to 1$. Now

$$\begin{aligned} F(i) = \infty &\implies ci = -d \\ F(-i) = 0 &\implies ai = b \\ F(1) = 1 &\implies a + b = c + d. \end{aligned}$$

Therefore $F(z) = -i\frac{z+i}{z-i}$. Hence the inverse transformation S from $\mathbb{R} \cup \{\infty\}$ to $\{z : |z| = 1\}$ is

$$S(w) = \frac{w-i}{w+i}.$$

First observe that $S(-1) = i$, $S(0) = -1$ and $S(1) = -i$, i.e., the real-axis goes to a circle by this transformation. Now putting $w = x + iy$ we have

$$\begin{aligned} S(w) &= \frac{w-i}{w+i} \\ &= \frac{x^2+y^2-1}{x^2+(y+1)^2} + i\frac{-2x}{x^2+(y+1)^2}. \end{aligned}$$

Therefore $|S(x)| = 1$ and $(0, 1)$ goes to $(0, 0)$. Hence the image of $\{x+iy : y > 0\}$ under this transformation is the unit disk i.e., $\{x + iy : x^2 + y^2 < 1\}$.

2. Find the harmonic conjugate of $u(x, y) = sinxcoshy$ vanishing at $(1, 0)$.
Answer: We see that $u_{xx}+u_{yy} = -sinxcoshy+sinxcoshy = 0$. Therefore $u(x, y)$ is a harmonic function. Let $v(x, y)$ be the conjugate harmonic of $u(x, y)$ such that $u + iv$ is analytic. Then by Cauchy-Riemann equations, we have $u_x = v_y$ and $u_y = -v_x$. Now

$$\begin{aligned} v(x,y) &= \int \frac{\partial u}{\partial x} dy + g(x) \\ &= cosxsinhy + g(x). \end{aligned}$$

Differentiating with respect to x, we get $\frac{\partial v}{\partial x} = -sinxsinhy + g^{'}(x) = -\frac{\partial u}{\partial y} = -sinxsinhy$. Therefore g is constant. But $v(1, 0) = 0$ gives $g(x) = 0$ for all x. Hence $v(x, y) = cosxsinhy$. This is the required result.

3. Give an example of a region and a function f in $H(\Omega)$ such that there is no power series convergent at all points of whose sum is $f(z)$.
Answer: Consider $f(z) = \frac{1}{z-1}$ for $z \in \mathbb{C}\backslash\{1\}$. Clearly this is an analytic function on this region. But there are two power series representations namely for $|z| < 1$, $f(z) = \sum_{n=0}^{\infty} z^n$ and for $|z| > 1$, $f(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n-1}}$.

4. If Ω is a region and f^2 and $\bar{f}$ are analytic in Ω show that f is necessarily a constant on Ω.

Answer: Let $f = u + iv$, where $u = u(x, y)$ and $v = v(x, y)$ be real-valued functions. Then $\bar{f} = u - iv$ which is analytic by hypothesis. Now $f^2 = u^2 - v^2 + i2uv$ and $\bar{f}^2 = u^2 - v^2 - i2uv$ both are analytic.

Therefore $f^2 + \bar{f}^2 = 2(u^2 - v^2)$ which is a real-valued analytic function and hence constant. Similarly uv is also a constant function. Thus f^2 and $\bar{f}^2$ are constant functions. Now using the fact that $\bar{f}$ is analytic, f has to be a constant function.

5. If $\gamma : [0,1] \to C$ is continuously differentiable show that $\int\limits_{\gamma} \frac{1}{\eta - z} d\eta \to 0$ as $z \to \infty$.

Answer: We have $\int\limits_{\gamma} \frac{1}{\eta - z} d\eta = \int\limits_0^1 \frac{1}{\gamma(t) - z} dt$.

Now $\left| \int\limits_0^1 \frac{1}{\gamma(t)-z} dt \right| \leq \sup \left|\frac{1}{\gamma(t)-z}\right| \int\limits_0^1 |\gamma'(t)| dt$. Since $\gamma([0,1])$ is bounded, $\sup \left|\frac{1}{\gamma(t)-z}\right| \to 0$ as $z \to \infty$.
Hence $\int\limits_{\gamma} \frac{1}{\eta - z} d\eta \to 0$ as $z \to \infty$.

6. Find the nature of singularity of the following functions at 0 :

$$a) \frac{Log(1+z)}{z^2}$$
$$b) \frac{1}{1-e^z}$$
$$c) z^2 sin(\frac{1}{z})$$

Answer: (a) : We see that for $|z| < 1$, $\frac{Log(1+z)}{z^2} = \frac{1}{z} - \frac{1}{2} + \frac{1}{3}z^2 - \ldots$.
Hence $z = 0$ is a simple pole.

(b) $1 - e^z = -z[1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots] = -z[1 + f(z)]$, where $f(z) = \frac{z}{2!} + \frac{z^2}{3!} + \ldots$. Clearly, $z = 0$ is a simple pole of $\frac{1}{1-e^z}$.

(c) The Laurent expansion at $z = 0$ of
$z^2 sin(\frac{1}{z}) = z - \frac{1}{3!z} + \frac{1}{5!z^3} - \ldots$. Therefore $z = 0$ is an essential singularity.

7. If f is a given entire function, find all entire functions g such that $|g(z)| \leq |f(z)|^2$ for all $z \in \mathbb{C}$.

Answer: Since $|g(z)| \leq |f(z)|^2 = |f^2(z)|$ for all z, the zeros of f^2 should be zeros of g. Consider $h(z) = \frac{g(z)}{f^2(z)}$, $z \in \mathbb{C}$. Clearly h is an entire function as g and f^2 are so. Also $|h(z)| \leq 1$. Now by Liouville's theorem h is constant. Therefore $g = cf^2$, where c is a constant with $|c| \leq 1$.