B. Math. Third Year
Second Semester - Analysis IV
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1. If X is a compact metric space, prove that $C(X)$ is a separable metric space.

Solutions: Since X is compact we can find countable $\{B_\delta(z_j)\}$ is dense in X. Define $f_j(x) = d(x, z_j)$. Let $\mathcal{M} \subset C(X)$ consist of functions which are finite product of f_j. Let $\mathcal{A}$ consist of function of the form $f = \sum_{k=1}^{N} a_k h_k, \quad a_k \in \mathbb{Q},\ h_k \in \mathcal{M}$. Now $\mathcal{A}$ is an algebra and its separate point and has non vanishing property. Stone-Weierstrass Theorem will give $\mathcal{A}$ is dense in $C(X)$. And it is not difficult to see that $\mathcal{A}$ is countable. □

2. If X is a compact metric space and $\mathcal{A}$ is a closed subalgebra of $C_{\mathbb{R}}(X)$ that separates points of X, prove that $\mathcal{A} = C_{\mathbb{R}}(X)$ or there is a $x_0 \in X$ such that $\mathcal{A} = \{f \in C_{\mathbb{R}}(X)) : f(x_0) = 0\}$.

Solutions: If $\mathcal{A}$ has unit then $\mathcal{A} = C_{\mathbb{R}}(X)$ as $\mathcal{A}$ is closed. Suppose $\mathcal{A}$ does not have unit. Let for $f \in \mathcal{A}$ there is $x_f \in X$ such that $f(x_f) \neq 0$ and $x_f \neq x_g$ for $f \neq g$. Then using compactness of X there exist $f_1, f_2, \cdots, f_n$ such that $X \subset \bigcup_{i=1}^{n} B_\delta(x_{f_i})$ the using continuity of f_i we get $g = \sum_{i=1}^{n} |f_i(x)|^2 \in \mathcal{A}$ is non vanishing. So $\mathcal{A}$ has unit as $\frac{1}{g}$ is in $\mathcal{A}$ so we get contradiction therefore we are done. □

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x, y) = (x^2 - y^2, 2xy)$. Prove that f is locally one-one but not one-one on $\mathbb{R}^2 \setminus (0, 0)$ and discuss inverse function theorem at $(1, 1)$.

Solutions: We have

$$\mathbf{f'(x, y)} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

Therefor $|f'(x, y)| = 4(x^2 + y^2) \neq 0$ when $(x, y) \neq (0, 0)$. There f is 1-1 in any nbd of a point in $\mathbb{R}^2 \setminus (0, 0)$. But $f(-2, 2) = (0, -8) = f(2, -2)$ so f is not globally 1-1.
Now $|f'(1, 1)| = 8$ so there exist a nbd U of $(1, 1)$ and nbd V of $(0, 2) = f(1, 1)$ such that $f(U) = V$ there exist $g : V \to u$ such that $g(f(x)) = x$ for all $x \in U$. □

4. Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function and $s_n(x)$ be the n-th partial sum of the Fourier series at $x \in \mathbb{R}$. Prove that for $x \in \mathbb{R}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} s_i(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2}\frac{\sin^2 nt}{\sin^2 t}dt. \tag{0.1}$$

Solutions: From Rudin page 189 equation (78) we have

$$s_i(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_i(t)dt, \tag{0.2}$$

where $D_i(t) = \frac{\sin(i+\frac{1}{2})t}{\sin\frac{t}{2}}$. Changing $-t$ to t in above integral together with the fact $D_i(t) = D_i(-t)$ we get

$$s_i(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)D_i(t)dt. \tag{0.3}$$

Now (0.2) and (0.3) will give

$$s_i(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2}D_i(t)dt. \tag{0.4}$$

Now

$$\sum_{i=0}^{n-1} D_i(x) = \frac{1}{2\sin^2\frac{t}{2}}\sum_{i=0}^{n-1} 2\sin(i+\frac{1}{2})t\sin\frac{t}{2} = \frac{1-\cos nt}{2\sin^2\frac{t}{2}} = \frac{\sin^2 nt}{\sin^2 t},$$

In above we use $2\sin a\sin b = \cos(a-b) - \cos(a+b)$ and $1-\cos a = 2\sin^2\frac{a}{2}$. So above together with (0.4) will give the result. □

5. Let $f(x) = 1$ if $|x| \leq 1$ and $f(x) = 0$ if $0 < |x| \leq \pi$ and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. ind the Fourier coefficients of f and deduce that $\sum_1^\infty \frac{\sin n}{n} = \frac{\pi-1}{2}$.

Solutions: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}$, $a_n = \int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\frac{\sin n}{n}$, $b_n = \int_{-\pi}^{\pi} f(x)\sin nx dx = 0$. Now we have

$$f(x) = \frac{a_0}{2} + \sum_n a_n\cos nx + \sum_n b_n\sin nx$$

At $x = 0$ we have the result

$$1 = \frac{1}{\pi} + \frac{2}{\pi}\sum \frac{\sin n}{n} \Rightarrow \frac{\pi - 1}{2} = \sum_n \frac{\sin n}{n}.$$

□

6. Show that the set of all polynomials of degree at most 3 with coefficients from $[1, 1]$ is compact in $C[0, 1]$. Does the result hold if coefficients are not assumed to be from $[1, 1]$.

Solution: Let $S = \{f : f(x) = a_0 + a_1x + a_2x^2 + a_3x^3,\ |x| \le 1\ and\ |a_i| \le 1\}$. Then $\|f\|_\infty \le 4$ and $|f'(x)| \le 6$ for all $f \in S$. So we have $|f(x) - f(y)| \le 6|x - y|$ for all $f \in S$. Now by arzella-ascolli we have the result. □

7. Prove that $\Omega = \{A \in L(\mathbb{R}^n) : detA \ne 0\}$ is open and $A \to\ A^{-1}$ is continuous on Ω.

Solution: We realize $A \in L(\mathbb{R}^n)$ as an element of $\mathbb{R}^{n^2}$ then det A is a polynomial therefore continuous. If We can realize $A^{-1} = \frac{adjugate\ of\ A}{detA}$, then we have the result. □

8. Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π periodic function s_n be the nth partial sum of the fourier series.
(a) If $s(x) \lim_{t\to 0} \frac{f(x+t)+f(x-t)}{2}$ exist show that $\frac{1}{n}\sum_{i=0}^{n-1} s_i(x) \to s(x)$.

Solutions: $K_n = \frac{1}{n}\sum_{i=0}^{n-1} D_i(x) = \frac{1}{n}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}$ we can see that $K_n \ge 0$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$. $K_n(x) \le \frac{2}{n\sin^2\frac{\delta}{2}},\quad 0 < \delta < |x| \le \pi$. We have

$$\frac{1}{n}\sum_{i=0}^{n-1} s_i(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)dt \quad (see\ 0.1)$$

Now proceed as in Theorem 7.26 of rudin. □

(b) If f is differentiable such that $f' \in \mathcal{R}[-\pi, \pi]$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2dx \le 1$ then $|f(x) - s_n(x)| \le \frac{2}{\sqrt{n}}$ for all x and $n \ge 1$.

solutions: We have

$$f(x) - s_n(x) = \sum_{k=n+1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]$$

using Integration by parts we have

$$a_n = \frac{b'_n}{n} \quad and \quad b_n = \frac{a'_n}{n}$$

In above $a'_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos nx dx$ and $b'_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin nx dx$. Therefore we have

$$\begin{aligned} |f(x) - s_n(x)| &\leq \sum_{k=n+1}^{\infty} \frac{1}{k}[|a'_k| + |b'_k|] \\ &\leq 2\Big(\sum_{k=n+1}^{\infty} \frac{1}{k^2}\Big)^{\frac{1}{2}}\Big(\sum_{k=n+1}^{\infty} |a'_k|^2 + |b'_k|^2\Big)^{\frac{1}{2}} \end{aligned}$$

Now we have $\sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \int_n^{\infty} \frac{1}{x^2} = \frac{1}{n}$ and $\sum_k |a'_k|^2 + |b'_k|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(x)|^2 dx$ Then from above we have the result.