SEMESTRAL EXAMINATION
ANALYSIS IV, B. MATH II YEAR
II SEMESTER, 2012-2013

1. Let $f : (a, b) \to \mathbb{R}$ be continuous and $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2} \quad \forall\ x, y \in (a, b)$. show that f is convex.

solution: Let f is not conves then there exist $c,\ d \in (a, b)\ c < d$ and a $t \in (0, 1)$ such that

$$f(tc + (1-t)d) > tf(c) + (1-t)f(d) \quad (1).$$

Define $\varphi(x) = f(x) - \frac{f(d)-f(c)}{d-c}(x-c) - f(a)$. Then $\varphi(a) = \varphi(b) = 0$ we can calculate using (1) and show that

$$\varphi(tc + (1-t)d) > 0.$$

Therefor we have $\gamma = \sup_{x\in[c,d]} \varphi(x) > 0$, Let $x_0 = \inf\{u \in [c, d] : \varphi(u) = \gamma\}$, since φ is continuous $\varphi(x_0) = \gamma > 0$, $\exists\ h > 0$ such that $\varphi(c \pm h) > 0$ and $c \pm h \in (c, d)$. we also have $\varphi(\frac{x+y}{2}) \leq \frac{\varphi(x)+\varphi(y)}{2}$. Now

$$\varphi(x_0) \leq \frac{\varphi(c-h) + \varphi(c+h)}{2} < \frac{\varphi(c) + \varphi(c)}{2} = \varphi(c)$$

we have contradiction. □

2. Let $\{f_n\}$ be a sequence of maps from $\mathbb{R}$ to $\mathbb{R}$ which is equicontinuous and uniformly bounded. Prove that there is a subsequence $\{f_{n_j}\}$ which converges pointwise to a continuous function on $\mathbb{R}$.

Solution: Arzela-Ascoli theorem. □

3. Let $\{f_n\}$ be a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ which is pointwise bounded. Prove that for any $a < b$ the interval contains an open interval on which the sequence is uniformly bounded.

Solution: Let $A_m = \cap_n f_n^{-1}[-m, m]$ then A_m is closed. Since f_n are point wise bounded we have

$$\mathbb{R} = \cup_m A_m.$$

Now thanks to Baire-category one of A_m will contain a open interval so we are done. $\square$

4. Let $f(x) = |\sin x|$ write down the fourier series of and prove that it converges to f at every point at every point.

Solution: $|\sin x|$ is Lipschitz continuous, so by 8.14 Theorem of rudin fourier series converges. Compute a_n, b_n. $\square$

5. Let $f(c) = \sum_{n=0}^{\infty} a_n c^n$ for all $c \in \mathbb{C}$ with $|c| \leq 1$ where $\{a_n\}$ is sequence of complex numbers such that $\{n^2 a_n\}$ is bounded. Show that $f(c) = 0$ whenever $|c| = 1$ implies $f(c) = 0$ for all $c \in \mathbb{C}$ with $|c| \leq 1$.

solution: $|f(c)| < M \sum_n \frac{1}{n^2} < \infty$ for $|c| \leq 1$. Therefor f is analytic now use Uniqueness theorem. $\square$

6. Prove that $\cos = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{n \sin 2nx}{4n^2-1}$ if $0 < x < \pi$.

Use $2 \sin a \cos b = \sin(a+b) + \sin(a-b)$ and $\cos a \cos b = \cos(a-b) - \cos(a+b)$. Now we will get $a_n = 0,\ n > 0$ and

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx dx = \frac{2n}{\pi} \frac{1 + (-1)^n}{n^2 - 1}.$$

Hence the result.