MID-SEMESTER EXAMINATION
B. MATH III YEAR, II SEMESTER February 2016
ANALYSIS IV.

1. Let B be the space of all binary sequences. Define $m : B \times B \to \mathbb{R}$ by $m(b, b) = 0$ and if $b \neq c$, then $m(b, c)$ is $1/2^k$, where $k = \min\{j : \ b_j \neq c_j\}$, $b = (b_j)$ and $c = (c_j)$. (i) Show that m is a metric on B. (ii) Prove that B is separable. (iii) Prove or disprove that B is complete. (iv) Write an isometry from B to ℓ^∞.

Solution: (i) Metric properties of B can be verified easly. (ii) To see that B is separable, we need to produce a countable dense set in B. Choos D to be the set of binary sequences with finitely non-zero terms. This set is countable and dense.If $b = (b_1, b_2, ...)$ is a binary sequence and $\epsilon > 0$, choose N so that $1/2^N < \epsilon$. Then $m(b, b') < \epsilon$ where $b' = (b_1, b_2, \cdots, b_{N-1}, c_N, 0, 0, ...)$ belongs to D. Here c_N is chosen as the entry different from Nth entry of b. (iii) B is complete since every Cauchy sequence in B is eventually constant. (iv) Define the map $L : B \to \ell^\infty$ by

$$L\big((b_n)\big) = (b_n/2^n).$$

Here ℓ^∞, the space of all bounded complex numbers is equipped with sup metric.

2. Let $f : [0, 1] \to \mathbb{R}$ be a function. The graph of f is $\{(x, f(x)) : x \in [0, 1]\}$. Give a function f whose graph is closed, but not bounded in $\mathbb{R}^2$. Show that f is continuous on I if and only if the graph is closed and bounded.

Solution: The function $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$ has its graph closed, but not bounded. The other part is a standard result, see for example Simmons book or Royden.

3. Consider the functions g_n and g defined on $\mathbb{R}$ by $g(x) = x$ and $g_n(x) = x+1/n$. Prove that g_n converges to g uniformly on $\mathbb{R}$, but g_n^2 doesn't convege uniformly to g.

Solution: $g_n(x)-g(x) = 1/n$ which is a convergent sequence independent of x. On the other hand, $g_n^2(x) - g^2(x) = 2x/n + 1/n^2$ which cannot converge to 0 as n gooes bigger unifrlmy in x. To understand this, it is enough to produce (x_n) and some ϵ such that $|f_n(x_n) - f(x_n)| > \epsilon$ for all n. Choose for example, $x_n = n$ and $\epsilon = 2$.

4. Define $h_n : [-1, 1] \to \mathbb{R}$ by

$$h_n(x) = \begin{cases} -1, & \text{if } x < -1/n \\ nx, & \text{if } -1/n \le x \le 1/n \\ 1 & \text{if } x > 1/n. \end{cases} \tag{0.1}$$

Determine the family $\{h_n\}$ is equi-continous or not.

Solution: The given family is not equicontinuous at 0. Equicontinuity at 0 means that, for $\epsilon > 0$, there exists δ so that $|h_n(x) - 0| < \epsilon$ whenever $|x| < \delta$, for all n. From this it follows that if x_k converges to 0, then $h_n(x_k)$ converges to 0 as k goes to ∞, uniformly in n. But the given function does not satisfy this. For example, $x_k = \frac{1}{\sqrt{k}}$ goes to 0. But $h_k(x_k) = 1$ for all k.

For the last two problems, see Simmons or Royden.