1. Let $\mathcal{A}$ be a closed algebra of real continuous functions on a compact metric space $X$ that separates points of $X$ and nowhere vanishes on $X$.
(a) If $f, g \in \mathcal{A}$, prove that $|f|$ and $\max\{f, g\}$ are in $\mathcal{A}$.
(b) For $f \in C_{\mathbb{R}}(X), x \in X$ and $\epsilon > 0$, prove that there is a $g \in \mathcal{A}$ such that $g(x) = f(x)$ *and* $g(y) > f(y) - \epsilon$ for all $y \in X$.

**Solution:** (a) Let $\{P_n\}$ be a sequence of polynomial (Weierstrass Approximation Theorem) $[0,1]$ such that $\lim_{n\to\infty} \|P_n(t) - \sqrt{t}\|_\infty = 0$ Now $g = f^2 \in \mathcal{A}$ define $h = \frac{g}{\|g\|}$ and $h_n = P_n(h) \in \mathcal{A}$. Now $\max_{x\in X} \|h_n(x) - \sqrt{h(x)}\| \to 0$ as $n \to \infty$. Now we get $\sqrt{h} = \frac{|f|}{\|f\|} \in \mathcal{A}$ imply $|f| \in \mathcal{A}$.
Now $\max\{f, g\} = \frac{f+g+|f-g|}{2} \in \mathcal{A}$.

(b) For each $y \in X \setminus \{x\}$ ( w.l.o.g we can assume $f(y) \neq f(x) \quad y \in X \setminus \{x\}$) define

$$\phi_y(t) = f(y)\frac{f(x) - f(t)}{f(x) - f(y)} + f(x)\frac{f(y) - f(t)}{f(y) - f(x)} \in \mathcal{A}.$$

Then $\phi_y(x) = f(x)$ and $\phi_y(y) = f(y)$. Now it is easy to see that

$$X \subset \bigcup_{y\in X} D_y, \quad \textit{where} \quad D_y = \{t \in X : f(t) - \epsilon < \phi_y(t) < f(t) + \epsilon\}$$

Since $X$ is compact then there exist finite $y_1, y_2, \cdots, y_n$ such that

$$X \subset \bigcup_{i=1}^{n} D_{y_i}$$

Now if we define $g = \frac{\phi_{y_1}+\phi_{y_2}+\cdots+\phi_{y_n}}{n}$ we get the result. $\square$

2. Let $\Phi : C[0,1] \to C[0,1]$ is given by $\Phi(f)(x) = \int_0^x f(t)dt$.
(a) Prove that $\Phi$ is continuous and $\Phi(B)$ is relatively compact for any bounded set $B \subset C[0,1]$.
(b) Is $\Phi$ is contraction? Does $\Phi$ have a unique fixed point? Justify your answer.

**Solution:** (a) We can see that $|\Phi(f)(x)| \leq |x|\|f\|_\infty$ and using the linearity of $\Phi$ we get

$$\|\Phi(f - g)\|_\infty \leq \|f - g\|_\infty. \quad (0.1)$$ equi

Let $B \subset C[0,1]$ is bounded then $\|\Phi(f)\|_\infty \le \|f\|_\infty \le M \;\; \forall\, f \in B$. Now equi 0.1 will give the equicontinuity of $\big(\Phi(f)\big)_{f\in B}$. Now Arzela-Ascoli will give the result.

(b) Let $f(t) = 1$ then $\|\Phi(f)\|_\infty = \|f\|_\infty = 1$, so $\Phi$ is not a contraction.
Let $g$ is a fixed point of $\Phi$ then $\Phi(g) = g$ i.e $\int_0^x g(t)dt = g(x)$ on $[0,1]$. This will give $g$ is differentiable and $g'(x) = g(x)$ with $g(0) = 0$. This is only true if $g(x) = 0$ on $[0,1]$ therefore $\Phi$ has a unique fixed point. $\square$

3. Let $X$ be a complete metric space and $\phi : X \to X$ be a map.
(a) If $\phi$ is a contraction prove that $\phi$ has a unique fixed point $x \in X$ and $\lim_{n\to\infty} \phi^n(y) = y$ for all $y \in X$.
(b) Suppose there is a sequence $(a_n)$ such that $\sum_n a_n < \infty$ and $d(\phi^n(x), \phi^n(y)) \le a_n d(x,y)$ for all $n \ge 1$ and $x, y \in X$. Prove that $\phi$ has a unique fixed point $x \in X$ and $\lim_{n\to\infty} \phi^n(y) = y$ for all $y \in X$.

**solution:** (a) Let $\phi(x) = x$ and $\phi(y) = y$ and $x \neq y$, since $\phi$ is contraction we have $d(x,y) = d(\phi(x), \phi(y)) \le cd(x,y) < d(x,y),\ c < 1$ this is not true so $x = y$.
Let $y \in X$ then

$$d(\phi^n(y), x) = d(\phi^n(y), \phi(x)) \le cd(\phi^{n-1}(y), x) \le \cdots \le c^{n-1} d(\phi(y), x).$$

Now the result will follow from continuity of $d : X \times X \to \mathbb{R}$ and $c < 1$.

(b) Since $\sum a_n < \infty$ then $\lim_{n\to\infty} a_n = 0$ ,i.e $|a_n| < c < 1 \quad \forall\, n > M$. Now we observe that for $n > M$

$$d(\phi^n(x), \phi^n(y)) \le a_1 a_2 \cdots a_n d(x,y) < a_1 a_2 \cdots a_M c^{n-M} d(x,y).$$

Now the result will follow from above method. $\square$

4. (a) Discuss Implicit Function Theorem for $F$ at $(2,-1,2,1)$ where $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by $F(x,y,u,v) = (x^2 - y^2 - u^3 + v^2 + 4, 2xy + y^2 - 2u^2 + 3v^4 + 8)$
(b) Let $X$ be a compact metric space and $g$ be a continuous function on $\mathbb{C}$. Prove that $\phi : C(X) \longrightarrow C(X)$ defined by $\phi(f) = g \odot f$ is continuous.

**solution:** (a) We can see that $F(2,-1,2,1) = 0$ and

$$f'(2,-1,0,0) = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix}$$

is invertible so there exist nbd of $W \subset \mathbb{R}^2$ and differentiable function $g : W \to \mathbb{R}^2$ such that $g(2,1) = (2,-1)$ and $f(g(y), y) = 0 \ \ \forall \ y \in W$.

(b)Let $f_n \to f$ in $C(X)$ as $n \to \infty$ i.e $\sup_{x\in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$. W.l.o.g we can assume $f_n(X)$ and $f(X)$ are contained in $B$,a compact subset of $\mathbb{C}$ for large enough $n$. Now $g$ is uniformly continuous on $B$. The continuity of $\phi$ will follow from the uniform continuity of $\phi$ and the following

$$\|\phi(f_n) - \phi(f)\|_\infty = \sup_{x\in X} |g(f_n(x)) - g((f(x))|.$$

$\square$

5. (a) Let $f$ be a continuously differentiable map of an open set $E$ of $\mathbb{R}^n$ into $\mathbb{R}^n$ . If $f'(x)$ is invertible for every $x \in E$, prove that $f$ is an open map.
(b) Suppose $f$ is a differentiable $2\pi$-periodic function such that $f' \in \mathcal{R}[-\pi, \pi]$. Assume $f \sim \sum c_n e^{inx}$ . Prove that $\sum n^2 |c_n|^2$ and $\sum |c_n|$ is finite.

**solution:** (a) See theorem 9.25 rudin (principle of mathematical analysis).
(b) Since $f$ is differentiable we can compute explicitely the fourier cofficient of $f'$ and write $f' \sim \sum inc_n e^{inx}$. Now we are given that $f'$ is Riemann integrable using parseval inequality we have

$$\sum n^2 |c_n|^2 = \int |f'(x)|^2 dx.$$

$\square$

6. Let $f \in \mathcal{R}[-2\pi, 2\pi]$ be a $2\pi$-periodic function and $f \sim \sum_{-\infty}^{\infty} c_n e^{inx}$.
(a) If for some $x \in [-\pi, \pi]$, there is a $\delta > 0$ and $M < \infty$ such that for all $t \in (-\delta, \delta)$, $|f(x+t) - f(t)| \le M|t|$, prove that $\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx} = f(x)$.
(b) prove that $\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - \sum_{n=-N}^{N} c_n e^{int}|^2 dt = 0$.

**Solution:** (a) See 8.14 Theorem of Rudin (principle of mathematical analysis). (b) 8.16 Parsevals theorem Rudin (principle of mathematical analysis). $\square$

7. Let $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$. Prove that $f(x) = \frac{\pi^2}{3} + 4 \sum_{1}^{\infty} \frac{\cos nx}{n^2}$

**solution:** We can write

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum_n b_n \sin nx$$

An explicite calculation will give

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{4\pi}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0$$

Hence the result.