BMath-II-Topology (Final Test)

Instructions: Total time 3 Hours. **Attempt as many question as you please, for a max score of 50**. You may use results proved in the class without proof. Use concepts, notations, terminology, results, as covered in the course. If you wish to use a problem from a homework/assignment as a result, supply its solution too.

1. Prove that the sphere $\mathrm{S}^n \subset \mathbb{R}^{n+1}$ is path connected for $n \geq 1$. (5)

2. Let X be a connected regular topological space having at least two points. Prove that X is uncountable. (5)

3. **(i)** Let X, Y be topological spaces, Y connected and $p : X \longrightarrow Y$ be a quotient map. Assume all fibers $p^{-1}(y),\ y \in Y$, are connected. Prove that X is connected.
   (ii) Does the conclusion of **(i)** hold if we drop the hypothesis of fibers of p being connected? Explain. (5+5)

4. Prove that the space obtained by identifying the boundary circle of a Möbius band to a point is normal. (10)

5. Let $n \geq 1$ and $f : \mathrm{S}^n \longrightarrow \mathbb{R}$ be continuous. Prove that there exists $x \in \mathrm{S}^n$ such that $f(x) = f(-x)$. (10)

6. Let $n \geq 1$ and $P(z_1, \cdots, z_n) \in \mathbb{C}[z_1, \cdots, z_n]$ be a polynomial. Prove that $\mathbb{C}^n$-$\mathrm{Z}(P)$ is path connected. Here $\mathrm{Z}(P) \subset \mathbb{C}^n$ is the set of all roots of P in $\mathbb{C}^n$ and $\mathbb{C}^n$ has the Euclidean topology. (10)

7. Prove that $\mathrm{GL}_n(\mathbb{C})$ is connected. (10)

8. Let $X \subset \mathbb{R}^2$ be the subspace defined by $X = \cup_{n=1}^{\infty} X_n$ where $X_n = \{(x, y) \in \mathbb{R}^2 | (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$ and $X' \subset \mathbb{R}^2$ be the subspace $X' = \cup_{n=1}^{\infty} X'_n$ where $X'_n = \{(x + \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$. Let $Y = X \cup X'$. Prove that X is homeomorphic to Y. (10)

9. Prove that $\mathbb{R}\mathrm{P}^n \cong \mathrm{O}(n + 1)/(\mathrm{O}(n) \times \mathrm{O}(1))$. (10)