Indian Statistical Institute
Bangalore Centre
B.Math (Hons.) III Year 2015-2016
First Semester
Statistics III

Mid-semester Examination Date :10.09.15

Answer as many questions as possible. The maximum you can score is 60.
All symbols have their usual meaning, unless stated otherwise.
State clearly the results you use.

1. Consider a random vector $X = (X_1, \cdots X_p)'$.

   (a) Find the 'best predictor' of X_1 among

   (i) **all** functions and (ii) **all linear** functions of $X_2, \cdots X_p$.

   (b) Denote 'the best linear predictor' of X_1 obtained in (a) (ii) by $X_{1.2\cdots p}$. Let $R_{1.2\cdots p} = X_1 - X_{1.2\cdots p}$.

   (i) Find variance of $X_{1.2\cdots p}$.

   (ii) Show that $R_{1.2\cdots p}$ is uncorrelated with every $X_j, j = 2, \cdots p$.

   [(3 + 4) + (2 + 3) = 12]

2. (a) When is a random vector $X = (X_1, \cdots X_p)'$ said to follow multivariate normal distribution ?

   (b) Suppose X follows $N_p(\mu, \Sigma)$. Find the distribution of $Y = B(k \times p) + b(k \times 1)$.

   (c) Consider X of Q(b). Partition X as $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and μ and Σ accordingly.

   (i) Show that X_1 and X_2 are independent if and only if $\Sigma_{12} = 0$.

   (ii) Let $Y = X_1 + MX_2$. Assume that Σ_{22} is p.d. Show that Y is independent of X_2 if and only if $M = -\Sigma_{12}\Sigma_{22}^{-1}$.

   (iii) Assuming that Σ_{22} is p.d, find the conditional distribution of X_1, given $X_2 = t$.

   (d) Consider p random variables $X_1, X_2, \cdots X_p$. Fill in the blank in the following statement with justification.

   "When the joint distribution of $X_1, X_2, \cdots X_p$ is - -, 'the best predictor' of X_1, based on $X_2, \cdots X_p$ coincides with the best predictor among **all linear** functions of $X_2, \cdots X_p$".

   [1 + 3 + (3 + 3 + 5) + 5 = 20]

3. (a) Define generalized (g-)inverse of a matrix.

   (b)For an $m \times n$ matrix A show the following.

   (i) The column space of A is the same as that of AA'.

   (ii) $A'(AA')^-$ is a g-inverse of A. [1 + (3 + 3) =7]

4. Consider the linear model

$$Y(n \times 1) = X(n \times p)\, \beta(p \times 1) + \varepsilon(n \times 1).$$

Here $E(\varepsilon) = 0$ and $Cov(\varepsilon) = \sigma^2 I_n$.

(a) Suppose l is in R^p. When is $l'\beta$ said to be estimable? Obtain the condition on l in terms of X matrix so that $l'\beta$ is estimable.

(b) How does one find a least square estimate $(\hat{\beta})$ of β ? Is it always unique ?

(c) Suppose $l'\beta$ is estimable. Show that $l'\hat{\beta}$ is always unique.

(d) Define residual sum of squares (R_0^2). Show that it can be expressed as $Y'QY$, where Q is a symmetric and idempotent matrix.

(e) Assume ε to be normally distributed. Suppose $l'\beta$ is estimable. Show that $l'\hat{\beta}$ and R_0^2 are independent.

[(1 + 2) + (4 + 1) + 3 + (1 + 4) + 6 = 22]

5. Suppose $X_i,\ i = 1, 2, \cdots n$ are i.i.d. standard normal variables. Let $X = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}'$.

(a) Suppose $Q_1 = X'AX$ and $Q_2 = X'BX$. If Q_1 and Q_2 follow χ^2 distributions with a and b degrees of freedom respectively and $A - B$ is non-negative definite, then show that $Q_1 - Q_2$ also follows χ^2 distribution with $a - b$ degrees of freedom.

(b) Consider the quadratic forms $Q_1 = X'AX$ and $Q_2 = X'BX$. Prove that if $AB = 0$ then Q_1 and Q_2 are independently distributed.

[6 + 4 = 10]