INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2014-15
Statistics - III, Midterm Examination, September 9, 2014

1. Suppose $Z_1, \ldots, Z_5$ are i.i.d. $N(0, \sigma^2)$. Let $\mathbf{X} = (X_1, \ldots, X_5)'$ where $X_1 = Z_1$ and $X_{i+1} = X_i + Z_{i+1}$ for $1 \leq i \leq 4$.
(a) Find the probability distribution of $\mathbf{X}$.
(b) Find $E(\mathbf{X}'A\mathbf{X})$ where $A = \mathbf{1}\mathbf{1}'$.
(c) Find the probability distribution of $(X_5 - X_3)^2 + (X_3 - X_1)^2$. [10]

2. Consider the model $\mathbf{Y} = \mathbf{X}\beta + \epsilon$, where $\mathbf{X}_{n \times p}$ has $\mathbf{1}$ as its first column and may not have full column rank; also $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$. Let $\hat{\beta} = (\mathbf{X}'\mathbf{X})^- \mathbf{X}'\mathbf{Y}$ and $RSS = (\mathbf{Y} - \mathbf{X}\hat{\beta})'(\mathbf{Y} - \mathbf{X}\hat{\beta})$, where $(\mathbf{X}'\mathbf{X})^-$ is any generalized inverse of $(\mathbf{X}'\mathbf{X})$.
Find the joint distribution of $(\frac{1}{n}\sum_{i=1}^n y_i, RSS)$. [10]

3. Consider the following model:
$y_1 = \alpha - \delta + \epsilon_1$
$y_2 = \delta - \gamma + \epsilon_2$
$y_3 = \alpha - \gamma + \epsilon_3$
$y_4 = -\alpha + \delta + \epsilon_4,$
where α, δ, γ are unknown constants and ϵ_i are uncorrelated random variables having mean 0 and variance σ^2.
(a) Show that $\alpha - 2\delta + \gamma$ is estimable. What is its BLUE?
(b) Does there exist a BLUE for $\alpha + \gamma$? Justify.
(c) Find an unbiased estimate of σ^2. [15]

4. Suppose $\mathbf{X} \sim N_n(\mathbf{0}, \sigma^2 I_n)$; A_i, $1 \leq i \leq p$ are symmetric $n \times n$ matrices of rank k_i, and $A = \sum_{i=1}^p A_i$ has rank k. Then show that
(i) $\mathbf{X}'A_i\mathbf{X} \sim \chi^2_{k_i}$, (ii) $\mathbf{X}'A_i\mathbf{X}$ are pairwise independent and (iii) $\mathbf{X}'A\mathbf{X} \sim \chi^2_k$ if and only if any two of the following are true:
(a) A_i are idempotent $\forall i$, (b) $A_iA_j = 0$, $i \neq j$, (c) A is idempotent.
You may use standard results from normal theory and matrix algebra by stating them without proof. [15]