Indian Statistical Institute, Bangalore
B. Math (III)
First Semester 2013-2014
End-Semester Examination : Statistics (III)

Date: 6-11-2013 **Maximum Score 60** **Duration: 3 Hours**

1. (a) For an $m \times n$ matrix A, define minimum norm *g-inverse.*

   (b) Prove that the following two statements are equivalent:

   i) G is a minimum norm *g-inverse* of A.

   ii) For any $\boldsymbol{y} \in \mathcal{C}(\boldsymbol{A})$, the column space of $\boldsymbol{A}$, $\boldsymbol{x} = \boldsymbol{G}\boldsymbol{y}$ is a minimum norm solution to the system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$.

   $[2 + 8 = 10]$

2. Consider the following generalized linear model (GLM)

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \\ E(\boldsymbol{\varepsilon}) &= \boldsymbol{0} \text{ and } Var(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{I}_n. \end{aligned}$$

   (a) Write down Γ, the class of all solutions to the normal equations $\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{Y}$.

   (b) Define estimability of a parametric function $\boldsymbol{l}'\boldsymbol{\beta}$.

   (c) Show that $\boldsymbol{l}'\boldsymbol{\beta}$ is estimable iff $\boldsymbol{l}'\boldsymbol{\gamma}$ is linear in $\boldsymbol{Y}$ and is same for all $\boldsymbol{\gamma} \in \Gamma$.

   (d) Show that $\boldsymbol{l}'\boldsymbol{\beta}$ is estimable iff $\boldsymbol{l}'(\boldsymbol{I} - \boldsymbol{K}) = \boldsymbol{0}$ where $\boldsymbol{K} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{X}$.

   $[3 + 2 + 8 + 7 = 20]$

3. Consider the following model

$$\begin{aligned} Y_1 &= \theta_2 - \theta_1 + \varepsilon_1 \\ Y_2 &= 2\theta_2 + \varepsilon_2 \\ Y_3 &= \theta_1 + \theta_2 + \varepsilon_3, \end{aligned}$$

   where the errors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are *iid* $\mathcal{N}(0, \sigma^2)$.

   (a) Is $\theta_2 - \theta_1$ estimable?

   (b) Derive a test for

$$H_0 : \theta_2 - \theta_1 = 0 \;Versus\; H_1 : \theta_2 - \theta_1 \neq 0. \qquad (1)$$

   (c) For $\boldsymbol{y}' = (2, 1, 3)$ carry out a suitable test for the testing problem in (1).

   $[2 + 8 + 8 = 18]$

$[PTO]$

4. Consider the following linear regression model

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{X\beta} + \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} &\backsim \mathcal{N}\left(\mathbf{0}, \sigma^2 \boldsymbol{I}_n\right) \end{aligned}$$

Let us partition the full column rank matrix $\boldsymbol{X}$ as $(\boldsymbol{X}_1, \boldsymbol{X}_2)$, and $\boldsymbol{\beta}'$ as $(\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')$ conformally. Derive a test for

$$H_0 : \boldsymbol{\beta}_2 = \mathbf{0} \; Versus \; H_1 : \boldsymbol{\beta}_2 \neq \mathbf{0}$$

using the notion of extra sum of squares.

[10]

5. Let us consider $(1+k) \times 1$ random vector $\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix}$ with a *positive definite* variance covariance matrix $\boldsymbol{V} = \begin{pmatrix} \sigma_y^2 & \boldsymbol{\delta}' \\ \boldsymbol{\delta} & \boldsymbol{\Sigma} \end{pmatrix}$. Here $Var(Y) = \sigma_y^2$, and $\boldsymbol{X}$ is a $k \times 1$ random vector with variance covariance matrix $\boldsymbol{\Sigma}$. $\boldsymbol{X}' = (X_1, X_2, \cdots, X_k)$.

(a) Obtain $\rho^2_{Y(X_1, X_2, \cdots, X_k)}$, $\rho_{Y(X_1, X_2, \cdots, X_k)}$ being the *multiple correlation coefficient* between Y and $(X_1, X_2, \cdots, X_k)$.

(b) Let w_{11} be the $(1,1)$ element of $\boldsymbol{V}^{-1}$ then express $\rho^2_{Y(X_1, X_2, \cdots, X_k)}$ in terms of w_{11}.

[6 + 6 = 12]