Second semestral backpaper exam 2023
Rings and Modules
B.Math. (Hons.) IInd year
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Q 1. Let P be a prime ideal and suppose $P \supset I_1 \cap I_2 \cap \cdots \cap I_n$ for some ideals I_j's. Then, prove that $P \supset I_j$ for some j.

OR

Q 1. Let $\theta : \mathbf{C}[X, Y] \to \mathbf{C}[T]$ be the ring homomorphism given by $X \mapsto T^2, Y \mapsto T^3$. Prove that Ker $\theta = (X^3 - Y^2)$.

Q 2. Let M be a left R-module over a commutative ring with unity, and let N be a submodule. If N and M/N are finitely generated, prove that M is finitely generated. Further, give an example of a free module over $\mathbb{Z}$ which has two minimal spanning sets of different cardinalities.

OR

Q 2. Define the Jacobson radical $Jac(R)$ of a commutative ring R with unity and prove that $x \in Jac(R)$ if and only if $1 + xy$ is a unit for all $y \in R$.

Q 3. Let A be a Noetherian ring. Show that every ideal of A contains a power of its radical.

OR

Q 3. Prove that a Boolean ring must be commutative, and of characteristic 2. Show further that every finitely generated ideal in a Boolean ring, must be principal.

Q 4. Let F be any finite field. Prove that there exists $f \in F[X, Y]$ such that

$$\{(x, y) \in F^2 : f(x, y) = 0\} = \{(0, 0)\}.$$

OR

Q 4. Suppose $d > 2$ is a square-free integer. Prove that $\mathbf{Z}[\sqrt{-d}]$ is not a UFD.

Q 5. Define the companion matrix $C(f)$ of a monic polynomial $f \in K[X]$ for a field K. Prove that its characteristic polynomial is f. Further, if $\deg f = 2$, prove that $C(f)$ is conjugate to its transpose by a matrix in $GL_2(K)$.

OR

Q 5. Let A be a commutative ring with unity. Prove that any set of n generators for the A-module A^n must be a basis.