Indian Statistical Institute

Final Examination 2022-2023

Analysis II, B.Math First Year

Time : 3 Hours   Date : 28.04.2023   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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You may freely apply any of the theorems covered in class.

Q1: (10 marks) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that f is a constant function if and only if there exists a partition P of $[a, b]$ such that $U(f, P) = L(f, P)$.

Q2: (20 marks) Let $f : [0, \infty) \to [0, \infty)$ be a function. Suppose $f|_{[0,m]}$ is Riemann integrable on $[0, m]$ for all $m > 0$. Determine with justification whether the following statements are true or false.

(i) $\lim\limits_{x\to\infty} f(x) = 0 \Longrightarrow \displaystyle\int_0^\infty f < \infty$.

(ii) $\lim\limits_{x\to\infty} f(x) < \infty$ and $\displaystyle\int_0^\infty f < \infty \Longrightarrow \lim\limits_{x\to\infty} f(x) = 0$.

Q3: (20 marks) Determine with justification whether the following statement is true or false:

$$\int_1^2 \Big(\sum_{n=1}^\infty \frac{1}{(1+x)^n}\Big) x dx = \sum_{n=1}^\infty \int_1^2 \frac{x}{(1+x)^n} dx.$$

Q4: (20 marks) Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers, and let $N > 1$ be a natural number. Assume that

$$a_{n+N} = a_n \qquad (n \geq 0).$$

(i) Prove that $\displaystyle\sum_{n=0}^\infty a_n x^n$ converges absolutely on $(-1, 1)$. (ii) Find a simplified closed-form expression for the function

$$f(x) = \sum_{n=0}^\infty a_n x^n \text{ on } (-1, 1).$$

Q5: (20 marks) Prove that $\displaystyle\sum_{m,n=1}^\infty \frac{(-1)^{m+n}}{mn}$ is conditionally convergent.

Q6: (20 marks) The moments of a continuous function f on $[0, 1]$ are given by

$$a_n(f) := \int_0^1 x^n f(x)\, dx \qquad (n \geq 0).$$

Suppose f and g are continuous functions on $[0, 1]$. Prove that $f = g$ on $[0, 1]$ if and only if

$$a_n(f) = a_n(g) \qquad (n \geq 0).$$