Indian Statistical Institute

Final Examination 2021-2022

Analysis II, B.Math First Year

Time : 3 Hours   Date : 26.05.2022   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) You may freely use any theorems that we have discussed in class.

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(1) Let $f : [a, b] \to \mathbb{R}$ be a bounded function such that f^2 is Riemann integrable on $[a, b]$. Does it follow that f is Riemann integrable on $[a, b]$?

(2) (15 marks) Prove that f is uniformly continuous on $\mathbb{R}$, where

$$f(x) = \int_0^x \frac{1}{1+t^{2022}}\,dt \qquad (\forall x \in \mathbb{R}).$$

(3) (15 marks) Prove that $\int_0^\infty \frac{\sin x}{x}\,dx$ is conditionally convergent.

(4) (15 marks) Let f be a uniformly continuous function on $\mathbb{R}$. For each $n \geq 1$, define

$$f_n(x) = f(x + n^{-1}) \qquad (\forall x \in \mathbb{R}).$$

Prove that $\{f_n\}_{n\geq 1}$ is uniformly convergent on $\mathbb{R}$.

(5) (15 marks) Prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{(nx+1)(nx-x+1)},$$

converges pointwise on $[0, 1]$, but it does not converge uniformly on $[0, 1]$.

(6) (15 marks) Suppose $a_{2n} = 1$ and $a_{2n+1} = 2$ for all $n \geq 0$. Consider the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Determine the domain of f and an explicit formula for it.

(7) (15 marks) Let f be a continuous function on $[0, 1]$. If

$$\int_0^1 x^n f(x)\,dx = 0,$$

for all $n \geq 0$, then prove that $f \equiv 0$ on $[0, 1]$.