Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis -I

Back Paper Examination | Date : 24 January 2022
Maximum marks: 100 | Time: 3 hours
Instructor: B V Rajarama Bhat

(1) Let D be the set of all sequences $\{a_n\}_{n\in\mathbb{N}}$ where $a_n \in \{1,2,3\}$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n$ exists. Show that D is countable. [15]

(2) Suppose $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are convergent sequences of real numbers and
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$
Show that the sequence $\{z_n\}_{n\in\mathbb{N}}$ defined by
$$z_n = \begin{cases} x_n & \text{if } n \text{ is odd;} \\ y_n & \text{if } n \text{ is even.} \end{cases}$$
is convergent and
$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n.$$
[15]

(3) Determine limsup and liminf of the following sequences of real numbers.

(i)
$$\{5 + (-\frac{1}{2})^n.7 : n \in \mathbb{N}\}.$$
(ii)
$$\{\frac{n}{4^n} - \frac{n}{5^n} : n \in \mathbb{N}\}.$$
(iii)
$$\{3 - \frac{2n-1}{n^2+1} : n \in \mathbb{N}\}.$$
[15]

(4) Show that the function $g : (0,2) \to \mathbb{R}$ defined by
$$g(x) = \frac{3}{x},\ x \in (0,2)$$
is not uniformly continuous. [15]

(5) Suppose $h : [0,1] \to [0,1]$ is a continuous function satisfying
$$h(t)h(1-t) \leq 0,\ \forall t \in [0, \frac{1}{2}).$$
Show that $h(\frac{1}{2}) = 0$. [15]

(6) Suppose $u : [-1,1] \to \mathbb{R}$ and $v : [-1,1] \to \mathbb{R}$ are functions, and $m : [-1,1] \to \mathbb{R}$ is defined by
$$m(x) = \max\{u(x), v(x)\},\ x \in [-1,1]$$
(i) Show that if u, v are continuous then m is continuous. (ii) Give an example to show that u, v are differentiable does not imply that m is differentiable. [15]

(7) Suppose $f : [0,1] \to \mathbb{R}$ is a non-constant differentiable function and $f(0) = f(1)$. Show that there exist s, t in $(0,1)$ such that $f'(s) < 0$ and $f'(t) > 0$. [15]