Indian Statistical Institute, Bangalore

B. Math (Hons.) First Year

First Semester - Analysis I

Semester Exam — Date: 15th November 2023

Maximum marks: 50 — Duration: 3 hours

Section 1: Answer any four and each question carries 6 marks

1. If (a_n) and (b_n) converge to a and b respectively, prove that $a_n - b_n \to a - b$ and $\max\{a_n, b_n\} \to \max\{a, b\}$.

2. Let (a_n) and (b_n) be sequences of non-negative reals. If $\sum a_n$ converges and $\sum b_n$ diverges, prove that $\sum \frac{\sqrt{a_n}}{n}$ converges and $\sum \frac{b_n}{1+b_n}$ diverges.

3. Let A be a nonempty subset of $\mathbb{R}$. Define f on $\mathbb{R}$ by $f(x) = \inf_{a\in A} |x - a|$. Prove that f is continuous on $\mathbb{R}$.

4. Prove that any continous function on $[a, b]$ with $(a < b)$ is uniformly continuous.

5. Let $f\colon [a, b] \to \mathbb{R}$ be continous and differentiable at $x \in [a, b]$. Let $g\colon [c, d] \to \mathbb{R}$ be a function such that $f([a, b]) \subset [c, d]$ and differentiable at $f(x)$. Then $h(t) = g(f(t))$ is differentiable at x.

6. Suppose f is differentiable on $\mathbb{R}$ such that f' is uniformly continuous on $\mathbb{R}$ and $\lim_{x\to\infty} f(x) = 0$. Prove that $\lim_{x\to\infty} f'(x) = 0$.

Section 2: Answer any two

1. (a) Prove that $\sum \frac{1}{n!}$ converges and $\lim(1 + 1/n)^n = \sum \frac{1}{n!}$ (**Marks: 7**).

   (b) Let $[x]$ be the largest integer less than or equal to x. Find points of continuity of $f(x) = [x]$ and $g(x) = [3x]$ on $(0, \infty)$. Justify your answer (**Marks: 6**).

2. (a) State and prove intermediate value property for continuous functions on intervals (**Marks: 7**).

   (b) Let f and g be bounded and uniformly continous. Prove that fg is uniformly continuous (**Marks: 6**).

3. (a) State and prove Taylor's Theorem (**Marks: 7**).

   (b) Let $f\colon (-1, 1) \to \mathbb{R}$ be differentiable at 0. If $-1 < a_n < 0 < b_n < 1$ with $b_n - a_n \to 0$, prove that $\frac{f(b_n)-f(a_n)}{b_n-a_n} \to f'(0)$ (**Marks: 6**).