Indian Statistical Institute, Bangalore Centre
B.Math. (III Year)/ M.Math. (II year) : 2014-2015
Semester I : Mid-Semestral Examination
Markov Chains

12.09.2014 Time: $2\frac{1}{2}$ hours. Maximum Marks : 80

Note: The paper carries 85 marks. Any score above 80 will be taken as 80.

1. (10+10 = 20 marks) Consider i.i.d. Bernoulli trials with probability p for success in each trial, where $0 < p < 1$. Let $X_0 = 0$; for $n = 1, 2, \cdots$ let $X_n = 0$ if n-th trial results in failure, and $X_n = k$ if $(n-k)$-th trial is a failure but j-th trial results in success for $j = (n-k)+1, (n-k)+2, \cdots, n-1, n$.

   (i) Find the transition probability matrix of $\{X_n\}$.

   (ii) Show that $\{X_n\}$ is recurrent.

2. (10 + 7 + 8 = 25 marks) (i) Let y be a transient state for a Markov chain $\{X_n : n \geq 0\}$ on a countable state space S. Let $G(x, y)$ denote the expected number of visits to state y with $X_0 = x$. Show that $G(x, y) < \infty$, for any $x \in S$.

   (ii) Let y be as in (i) above. Show that $\lim_{n\to\infty} P_{xy}^{(n)} = 0$, for any $x \in S$.

   (iii) Using the above, show that an irreducible Markov chain on a finite state space is recurrent.

3. (6 + 7 + 7 = 20 marks) (i) P is a transition probability matrix on a finite state space. Show that P^2 is also a transition probability matrix.

   (ii) If π is a stationary probability distribution for P, show that it is also a stationary probability distribution for P^2.

   (iii) Is the converse of (ii) above true?

4. (10 + 10 = 20 marks) Consider a Markov chain on a countable state space S with transition probability matrix P. Let $x, y \in S$ be fixed. Denote $\rho_{xy} = \text{Prob.}(T_y < \infty)$, where T_y is the first hitting time of state y. Show that

   (i) $P_x(T_y = n+1) = \sum_{z\neq y} P_{xz} P_z(T_y = n),\ n \geq 1$;

   (ii) $\rho_{xy} = P_{xy} + \sum_{z\neq y} P_{xz}\rho_{zy}$.