Indian Statistical Institute, Bangalore
B. Math.
Third Year, First Semester
Introduction to Stochastic Processes (probability Theory III)
Mid-term Examination Date : 17/09/09

Maximum marks: 100 Time: 3 hours

Note: Throughout Markov chains are assumed to be time homogeneous.

1. For the following transition matrices determine: (i) communicating classes; (ii) absorbing states ;(iii) recurrent states; (iv) transient states; (iv) positive recurrent states; (vi) null recurrent states:

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

[20]

2. Let $\{Y_n\}_{n\geq 0}$ be a Markov chain on a state space S with initial distribution π. For $n \geq 0$, define Z_n by $Z_n = Y_{2n+3}$. Then show that $\{Z_n\}_{n\geq 0}$ is a Markov chain. Determine its initial distribution and transition matrix. [10]

3. Let $\{C_n\}_{n\geq 0}$ be a Markov chain on state space $\{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For $i = 1, 2, 3, 4$ compute the expected time for absorption in state 4, that is, compute $h_i^4 = E(H^4/C_0 = i)$ where $H^4 = \inf\{n \geq 0 : C_n = 4\}$. Obtain a stationary measure for this transition matrix. [20]

4. Let $\{X_n\}_{n\geq 0}$ be a Markov Chain on a state space S. Show that for $i, j \in S$,

$$P(N_j = m/X_0 = i) = \begin{cases} \rho_{ij}\rho_{jj}^{(m-1)}(1-\rho_{jj}) & \text{if } m \geq 1 \\ 1-\rho_{ij} \text{ if } m = 0 \end{cases},$$

where and $N_j = \sharp\{n > 1 : X_n = j\}$ and

$$\rho_{ij} = P(X_n = j \text{ for some } n \geq 1/X_0 = i).$$

[15]

5. Show that a Markov chain on a finite state space has at least one recurrent state. [10]

6. Consider a game where there are N people and exactly one of them has a coin. In each round of the game the coin goes to another person at random. The movement of the coin clearly determines a Markov chain. Write down the transition probability matrix for this chain. For any person having the coin, compute the expected number of turns of the game for getting back the coin. [15]

7. Suppose $\{X_n\}_{n\geq 0}$ is a Markov process on a state space S and $i \in S$ is recurrent. Now for $j \in S$, if i leads to j, show that i communicates with j. [10]

8. Let Z be a random variable taking values in $I\!N = \{1, 2, 3, \ldots,\}$ with $P(Z = j) > 0$ for every j. Let $\{M_n\}_{n\geq 0}$ be a Markov chain on state space $I\!N$ with transition probability matrix P given by

$$P_{ij} = \begin{cases} P(Z = j) & \text{if } i = 1; \\ 1 & \text{if } j = i-1, i > 1; \\ 0 & \text{otherwise.} \end{cases}$$

Show that the Markov chain $\{M_n\}_{n\geq 0}$ is irreducible and recurrent. Further show that it is positive recurrent if and only if $E(Z) < \infty$. [15]