Indian Statistical Institute, Bangalore Centre
B.Math. (III Year) : 2010-2011
Semester I : Semestral Examination
Probability III (Stochastic Processes)

3.12.2010 Time: 3 hours. Maximum Marks : 100

Note: The paper carries 105 marks. Any score above 100 will be taken as 100. State clearly the results you are using in your answers.

1. [5+8+7=20 marks] Consider a Markov chain on $\{1, 2, 3\}$ having transition probability matrix $P = ((P_{ij}))$ with $P_{13} = P_{21} = 1, P_{31} = P_{32} = 0.5, P_{ij} = 0$ otherwise.

(i) Show that the Markov chain is irreducible.

(ii) Find the period.

(iii) Find the stationary distribution.

2. [10+10+10=30 marks] Consider i.i.d. Bernoulli trials with probability p for success in each trial, where $0 < p < 1$. Let $X_0 = 0$; for $n = 1, 2, \cdots$ let $X_n = 0$ if n-th trial results in failure, and $X_n = k$ if $(n-k)$-th trial is a failure but j-th trial results in success for $j = (n-k)+1, (n-k)+2, \cdots, n-1, n$. (So X_n denotes the length of success runs in Bernoulli trials.) Assume $\{X_n\}$ is a time-homogeneous Markov chain.

(i) Find the transition probability matrix.

(ii) Show that $\{X_n\}$ is recurrent.

(iii) Is $\{X_n\}$ positive recurrent?

3. [10 marks] Consider the simple branching chain with offspring distribution given by the discrete density function $f(\cdot)$. Assume that $f(1) < 1$. Show that any non-zero state is transient.

4. [10+8+7=25 marks] Let $\{N(t) : t \geq 0\}$ be a time-homogeneous Poisson process with rate $\lambda > 0$. For $n = 1, 2, \cdots$ let W_n be the waiting time until the n-th event.

(i) Show that $P(W_n < \infty) = 1$ for any n.

(ii) Find the distribution function and the probability density function of $W_n, n \geq 1$.

(iii) Let $0 \leq s < t$ and $n \geq 1$. Find $P(W_1 < s | N(t) = n)$.

5. [10 marks] Let $X(t) = \sum_{i=1}^{N(t)} Y_i, t \geq 0$ be a compound Poisson process (with the usual assumptions). Find $E(X(t))$ and $Var(X(t))$.

6. [10 marks] Show that the transition probability function of a birth and death process satisfies the system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt} P_{0j}(t) &= -\lambda_0 P_{0j}(t) + \lambda_0 P_{1j}(t), \quad t > 0, \\ \frac{d}{dt} P_{ij}(t) &= \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t), \quad t > 0, i \geq 1, \end{aligned}$$

with the initial condition $P_{ij}(0) = \delta_{ij}$, for any fixed $j \geq 0$. Here $\{\lambda_i : i \geq 0\}$ are the infinitesimal positive birth rates, $\{\mu_i : i \geq 1\}$ are the infinitesimal positive death rates.