Introduction to Stochastic Processes: B. Math (Hons.) III
Academic Year 2021-22, First Semester
Final Exam: Total Marks = 50
Duration: 3 Hours

Note:

- Please write your name on top of your answer booklet and sign a declaration that says *"I have not taken help from anyone (either online or in person) for solving the problems given in this exam".*
- This is an open-note examination. You are allowed to use your class notes. However, you are not allowed to talk to each other, consult any book or take any form of online help outside the course website.
- Notation and terminology are same as the ones used in the class.
- Please give full justification and quote appropriate result(s) covered in the class.
- Please email the solutions as one pdf file by 1:30 pm of January 07, 2022.

1. (10 marks) Using the definition of Markov chain as given in the class, show that for all $k, l \in \mathbb{N}$ and for all $i, i_1, i_2, \ldots, i_l \in S$,
$$\mathbb{P}_i\big(\tau_i(1) = k, X_{\tau_i(1)+1} = i_1, X_{\tau_i(1)+2} = i_2, \ldots, X_{\tau_i(1)+l} = i_l\big)$$
$$= \mathbb{P}_i(\tau_i(1) = k)\, \mathbb{P}_i\big(X_1 = i_1, X_2 = i_2, \ldots, X_l = i_l\big).$$

2. (7+3 = 10 marks) Suppose $\{X_n\}_{n\geq 0}$ is a Markov chain with state space S. For any subset $A \subseteq S$, define $cl(A)$ (the closure of A) to be the smallest closed subset of S containing A. Show that
$$cl(\{j\}) = \{k \in S : j \to k\}$$
for any state $j \in S$. If j is recurrent, then show that $cl(\{j\})$ is the communication class containing j.

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3. For each of the following Markov chains, we mention a state. Find, with justification, whether that state is transient or null recurrent or positive recurrent.

   (a) (5 marks) The state $(0, 0, \ldots, 0)$ in the simple random walk on the group $\mathbb{Z}_2^d$.

   (b) (5 marks) The state $(0, 0, 0)$ in the direct product of three simple random walks on $\mathbb{Z}$.

4. Suppose you have 52 cards with the letters $A, B, C, \ldots, Z$ and $a, b, c, \ldots, z$ written on them. Consider the following Markov chain on the set S of all permutations of the 52 cards. Start with any fixed arrangement of the cards and at each step, choose a letter at random and interchange the cards with the corresponding capital and small letters. For example if the letter "M/m" is chosen, then the cards "M" and "m" are interchanged. This process is repeated again and again.

   (a) (7 marks) Count, with justification, the number of communicating classes of this Markov chain.

   (b) (3 marks) Give a stationary distribution for the chain.

   (c) (3 marks) Is the stationary distribution unique? Justify your answer.

   (d) (7 marks) Find the expected number of steps for the Markov chain to return to its initial state.