Indian Statistical Institute, Bangalore
B.Math (Hons.) III Year/ M.Math II year 2018-2019
Semester I : Probability III/ Markov Chains

Final Exam | Date: 20.11.2018
Maximum Marks: 90 | Duration: 3 hours

Note: Any score above 90 will be taken as 90. State the results very clearly that you are using in your answers.

1. $(10+15)$ Let P be the transition probability matrix of an irreducible Markov chain on a state space of size n.
   (a) Show that $(I+P)^{n-1}$ is a positive matrix (a matrix with all entries positive).
   (b) If the chain is aperiodic and $n=3$, show that P^5 is also a positive matrix.

2. (15) Consider $i.i.d$ Bernoulli random variables $\{Z_i\}_{i=0}^{\infty}$ which take value 1 with probability p. Let $X_n = \left(\sum_{i=0}^{n} Z_i\right) \mod k$, for all $n \geq 0$ where k is some fixed integer. Show that $\{X_n\}$ is an irreducible homogeneous Markov chain on state space $\{0, 1, 2, \ldots, k-1\}$. Find its period and the limiting distribution of X_n.

3. $(15+5)$ Let $\{N(0,t] : t > 0\}$ be a time-homogeneous Poisson process with rate $\lambda > 0$. For $n \geq 1$, let S_n be the waiting time until the n-th event. Let $0 < s < t$ and $n \geq 1$.
   (a) Find $P\big[S_1 < s | N(0,t] = n\big]$
   (b) Prove that $\lim_{t\to\infty} \frac{N(0,t]}{t}$ exists and find the value of this limit.

4. (10) Let P be the transition probability matrix of an irreducible Markov chain on a finite state space. Each column sum of P is 1. Starting from any state i, find the expected time taken for the first return to i. (Hint: conditions for positive recurrence)

5. (10) Let $p_k = \frac{2}{3^{k+1}}$ $(k \geq 0)$. Find the fixed points of the function $f(s) = \sum_{k=0}^{\infty} p_k s^k$ in $[0,1]$ and the extinction probability of the Galton Watson process with offspring distribution p_k.

6. (15) Let $\{X_n\}_{n\geq 0}$ be a homogeneous Markov chain on state space $\mathbb{N}$ with transition probability matrix given by

$$P_{ij} = \begin{cases} p_j & \text{if } \ i = 1 \\ 1 & \text{if } \ j = i-1, i > 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $p_j > 0 \; \forall \; j \in \mathbb{N}$. Show that the Markov chain $\{X_n\}$ is irreducible and recurrent. Derive the necessary and sufficient conditions for the chain to be positive recurrent.

7. (10) For a continuous time pure birth process with infinitesimal birth rate λ, Find $P_{02}(t)$.