Indian Statistical Institute
B.Math. (Hons.) II & III Year
Second Semester Examination, 2005-2006
Stochastic Processes

Time: 3 hrs | Date:03-05-06 | Max. Marks : 100

1. Consider the success run chain with state space $S = \{0, 1, 2, \ldots\}$ and transition matrix $P = ((p_{ij}))$ where $p_{ij} = 0$ if $j \neq 0,\ i+1$; $p_{i0} = q_i$, $p_{i\ i+1} = p_i$. Show that the chain is irreducible and zero is a recurrent state iff $\sum_j q_j = \infty$. [10]

2. Suppose the Markov chain (X_n) with state space S is irreducible and recurrent. Fix $i \in S$.

   a) Describe explicitly the (unique) invariant measure $\gamma = \{\gamma_j,\ j \in S\}$ satisfying $\gamma_i = 1$. [5]

   b) Show that $0 < \gamma_j < \infty \quad \forall j \in S$. [5]

   c) Show that a unique stationary distribution exists iff $E_i\tau_i < \infty$ where $\tau_i = \inf\{k \geq 1 : \quad X_k = i\}$. [5]

3. Consider the Markov chain on $S = \{0, 1, 2, \ldots\}$ corresponding to the 'infinite capacity storage model' viz

$$X_{n+1} = (X_n + A_{n+1} - 1)^+$$

   for $n \geq 0$. Here $\{A_n\}$ are integer valued i.i.d random variables such that for each $n \geq 0$, $\quad A_{n+1}$ is independent of X_n. Let $A(s) = Es^{A_1}$, $\ 0 < s \leq 1$ be the generating function of A_1. Suppose $P(X_0 = k) = \pi_k,\ \ k \geq 0$ and (π_k) is a stationary distribution with generating function $\Pi(s)$. Show that if $0 < EA_1 < 1$ and $P(A_1 = 0) > 0$ then

$$\Pi(s) = \frac{(1 - EA_1)(1 - s)}{A(s) - s} \quad 0 < s < 1.$$

   [15]

4. Consider the Markov chain $(Z_n)_{n\geq 0}$ on $S = \{0, 1, \ldots\}$ corresponding to a branching process with immigration viz. $Z_0 = 1$

$$Z_{n+1} = I_{n+1} + \sum_{j=1}^{Z_n} Z_{nj}, \quad n \geq 0$$

where $\{Z_{nj}, n \geq 0, j \geq 1\}$ are i.i.d. non negative integer valued random variables and $\{I_n, n \geq 1\}$ are i.i.d integer valued random variables independent of $\{Z_{nj}, n \geq 0, j \geq 1\}$. Suppose I_1 has generating function $B(s)$ with $0 < \beta = B'(1) < \infty$ and Z_{01} has generating function $A(s)$ with $0 < \quad \alpha = A'(1) < 1$.

a) If $P = ((p_{ij}))$ is the transition matrix for $(Z_n)_{n\geq 0}$ show that for all $i \in S$,

$$\sum_{j=0}^{\infty} p_{ij} \cdot s^j = B(s)A(s)^i$$

[5]

b) Show that for $n \geq 0$,

$$E[Z_{n+1} \mid Z_n] = \beta + \alpha Z_n \quad \text{w.p.1.}$$

[5]

c) Show that if $\Pi(s)$ is the generating function for a stationary distribution for (Z_n) then $\Pi(s)$ satisfies

$$\Pi(s) = B(s)\Pi(A(s)) \quad 0 \leq s \leq 1.$$

[5]

d) Let $A(s) = q+ps \quad p+q = 1, \quad 0 < p < 1$, and $B(s) = e^{\lambda(s-1)}, \lambda > 0$. Compute $\Pi(s)$ (Hint : use (c)). [10]

5. Consider a Markov chain (X_n) on the state space $\{0, 1, \ldots, N\}$, with transition matrix $P = ((p_{ij}))$ consisting of 3 classes viz. $\{0\}, \{1, \ldots N-1\}$ and $\{N\}$, where 0 and N are absorbing states both accessible from any $k \in \{1, \ldots N-1\}$. Pick a reference state, say 1, and define a modified chain (Y_n) with transition matrix $Q = ((q_{ij}))$ with $q_{ij} = p_{ij} \quad i, j \neq 0, N \quad ; \quad q_{01} = q_{N1} = 1$.

a) Show that $\{Y_n\}$ is irreducible. [5]

b) Let $\tau_1 = \inf\{k \geq 1 \ : \ X_k = 0 \text{ or } N\}$. Show that $E_1\tau_1 = \frac{1}{\pi_0} + \frac{1}{\pi_N} - 1$ where $\pi = (\pi_k)$ is the stationary distribution for (Y_n). [10]

6. Let (X_t) be a continuous time Markov chain with state space S and parameters $Q = ((q_{ij}))$ and $\{\lambda(i) \ : \ i \in S\}$. Suppose $\lambda(i) \equiv \lambda, \quad \lambda > 0$ for all $i \in S$. Compute the transition probabilities $P_{ij}(t)$ explicitly in terms of Q and λ. [15]

7. Consider an $M/M/1$ queue with arrival and service rates corresponding to independent Poisson processes with parameters $a > 0$ and $b > 0$ respectively. Let $X_t =$ number in the system at time $t, \ t \geq 0$.

a) Compute the transition probabilities $Q = ((q_{ij}))$ and the holding time parameters $\{\lambda(i); \ i \geq 0\}$. [10]

b) Show that if $a < b$ then a unique stationary distribution exists. [10]