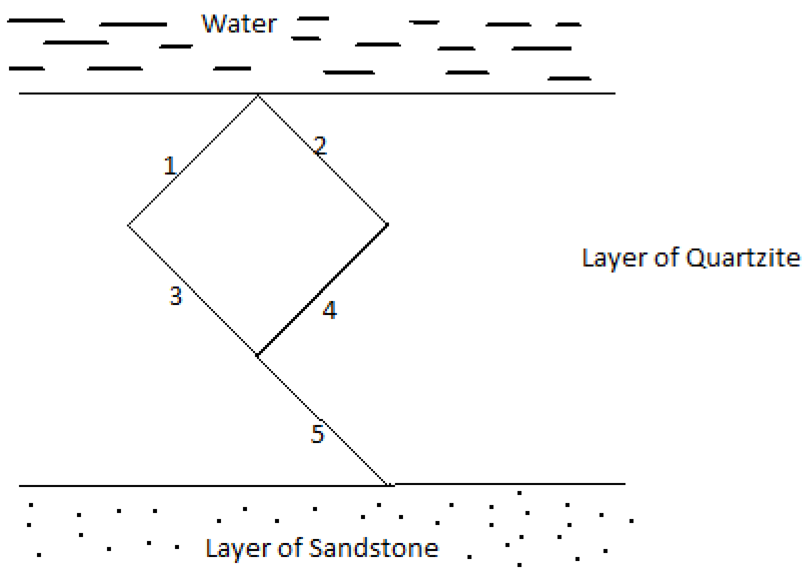
Probability II: B. Math (Hons.) I
Academic Year 2021-22, Second Semester
Final Exam: Total Marks = 50, Duration = 3 Hours

Note:

- Please write your name on top of your answer booklet.
- The exam is closed-book and closed-notes except the following: handwritten list of formulae, theorems etc. written on both sides of five A4-sized papers are allowed in the exam. If someone uses anything else (e.g., bigger paper, extra pages, printed material, etc.), that student's exam will be cancelled with a zero score in the final exam.
- It is absolutely important that you follow the rules mentioned above and the usual in-class examination rules or else, if caught, you will get a zero in the final exam. The teacher will also report against you to the appropriate authorities.

1. Consider the following schematic diagram of a drainage network model (as described in class), where each of the five paths is open with probability $p = 0.5$ and the paths behave independently of each other.



   (a) (10 marks) Let X be the number of open paths and Y be the indicator that water can pass through the layer of quartzite to the layer of sandstone. Find the conditional probability mass function of X given $Y = 1$.

   (b) (5 marks) Compute $E(X|Y = 1)$.

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2. A continuous random vector (X, Y) has a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} 2/3 & \text{if } x > 0, y > 0, x + y < 1, \\ c & \text{if } x < 1, y < 1, x + y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 marks) Compute c.

(b) (5 marks) Find the marginal probability density function of X.

3. (10 marks) Suppose $r(\geq 2)$ distinguishable umbrellas are distributed at random among $n(\geq 3)$ mathematicians. Let X be the number of mathematician(s) who get no umbrella and Y be the number of mathematician(s) who get exactly one umbrella. Show that

$$Cov(X,Y) = \frac{r(n-1)(n-2)^{r-1}}{n^{r-1}} - \frac{r(n-1)^{2r-1}}{n^{2r-2}}.$$

4. Suppose X and Y are jointly normal (i.e., bivariate normal) with $E(X) = E(Y) = 0$, $Var(X) = Var(Y) = 1$ and $Corr(X,Y) = \rho \in (-1, 1)$.

(a) (10 marks) Show that $(X,Y)^T \stackrel{d}{=} \left(Z,\ \rho Z + \sqrt{1-\rho^2}\, W\right)^T$, where $Z, W \stackrel{iid}{\sim} N(0,1)$.

(b) (5 marks) Using (a) or otherwise, state an algorithm to simulate the random vector $(X,Y)^T$ using $U, V \stackrel{iid}{\sim} Unif(0,1)$. Just state the method. No proof is required.