Midterm - Probability I (2024-25)
Time: 2.5 hours.
Attempt all questions. The total marks is 20.
You may quote any result proved in class without proof.

1. Let (Ω, P) be a probability space. Consider a sequence of events A_n such that $A_n \uparrow A$, that is $A_1 \subseteq A_2 \subseteq \cdots$ and $\cup_{n=1}^{\infty} A_n = A$. Show that $\lim_{n\to\infty} P(A_n) = P(A)$. **[4 marks]**

2. Suppose that n balls are randomly distributed into N compartments. Find the probability that exactly m balls will fall into the first compartment. Assume that all N^n arrangements are equally likely. **[4 marks]**

3. A closet contains n *pairs* of shoes. If $2r$ shoes are chosen at random (with $2r < n$), what is the probability that there will be *exactly* two complete pairs among them? **[4 marks]**

4. Urn A has 5 white and 7 black balls. Urn B has 3 white and 12 black balls. We flip a fair coin. If the outcome is heads, then a ball from urn A is selected, whereas if the outcome is tails, then a ball from urn B is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails given this information? **[4 marks]**

5. Recall that a Poisson(λ) random variable X has p.m.f. given by

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k \in \{0, 1, 2, \cdots\}.$$

Show that

$$E[X^n] = \lambda E\left[(X+1)^{n-1}\right],$$

and use this to compute $E[X^3]$. **[4 marks]**