

INDIAN STATISTICAL INSTITUTE
Probability Theory I: B. Math (Hons.) I
Semester I, Academic Year 2022-23
Final Exam

Full Marks: 50

Duration: 3 hours

- Show all your work and write explanations when needed. If you are using a result stated and/or proved in class, please quote it.
- You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.

1. (10 marks) Suppose $X \sim \text{Exp}(1)$ and $Y := 1 - 2e^{-X}$. Show that Y is an absolutely continuous random variable and find a probability density function of Y .
2. A random variable X is said to follow Laplace distribution with parameters $\mu \in \mathbb{R}$ and $\tau \in (0, \infty)$ (denoted by $X \sim \text{Laplace}(\mu, \tau)$) if X has a probability density function

$$f_X(x) = \frac{1}{2\tau} \exp\left(-\frac{|x - \mu|}{\tau}\right), \quad x \in \mathbb{R}.$$

- (a) (6 marks) Write down, with proper justification, an algorithm to simulate a random variable $Z \sim \text{Laplace}(0, 1)$.
- (b) (6 marks) If $X \sim \text{Laplace}(\mu, \tau)$, then what distribution does

$$W := \frac{|X - \mu|}{\tau}$$

follow? Justify your answer.

- (c) (3 marks) Using (b), compute $E(W)$.
3. (10 marks) Roads A and B are the only escape routes from a prison. Prison records show that of the prisoners who tried to escape, 40% used road A, and 60% used road B. The records also show that 80% of those who tried to escape via A, and 70% of those who tried to escape via B were captured. Suppose that two prisoners have independently and successfully escaped from the prison. What is the conditional probability that they used the same road to escape?
4. (15 marks) Fix two positive integers r_1, r_2 . Suppose r_1 many α 's and r_2 many β 's are arranged at random. Let X be the number of α -runs and Y be the number of β -runs. Find, with justification, the joint probability mass function of X and Y .