B3 - Probability III : Back-Paper Exam

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26th December, 2025. **Time Limit : 3 hours.** **Max. points : 50.**

All questions carry 10 points. Attempt all questions.

1. Let $X_1, \ldots, X_n, \ldots$ be Gaussian random variables (not necessarily independent or identically distributed) with mean μ and variance factor σ^2. Show that a.s.
$$\limsup_{n\to\infty} \frac{\max\{X_1, \ldots, X_n\} - \mu}{\sqrt{\log n}} \leq 2\sigma^2.$$

2. Let X be a non-negative integer valued random variable. Show that X is a Poisson(λ) random variable iff $\mathbb{E}(Xf(X)) = \lambda\mathbb{E}(f(X+1))$ for any bounded function $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$.

3. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables. Compute the following probabilities in each of the two cases : $\mathbb{E}(|X_1|) = \infty$ and $\mathbb{E}(|X_1|) < \infty$.
$$\mathbb{P}(\limsup \frac{|X_n|}{n} \leq 2025) \quad \text{and} \quad \mathbb{P}(\limsup \frac{|X_1 + \ldots + X_n|}{n} = \infty).$$

4. Let X_n, X be random variables. If $X_n \Rightarrow X$, show that the following hold.
   (a) For all open sets G, $\liminf \mathbb{P}(X_n \in G) \geq \mathbb{P}(X \in G)$.
   (b) For all compact sets K, $\limsup \mathbb{P}(X_n \in K) \leq \mathbb{P}(X \in K)$.
   (c) For all bounded Borel sets A such that $\mathbb{P}(X \in \partial A) = 0$, $\mathbb{P}(X_n \in A) = \mathbb{P}(X \in A)$. Here ∂A denotes the boundary of set A.

   Show that the inequalities in the first two parts may be strict.

5. (a) Let X be an integrable random variable and $\mathcal{F}$ is a trivial σ-field i.e., all events in $\mathcal{F}$ have probability 0 or 1. Show that $\mathbb{E}(X \mid \mathcal{F}) = \mathbb{E}(X)$. **(5)**
   (b) Let X, Y be random variables and U be an independent uniform $[0,1]$ random variable. Show that there exists a measurable function $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that $(X, Y) \stackrel{d}{=} (X, f(X, U))$. **(5)**

Notations

- $\mathbb{N}$ - set of Natural numbers; $\mathbb{Q}$ - set of Rationals ; $\mathbb{R}$ - set of real numbers. $\mathcal{B}$ - Borel σ-algebra.
- pmf - probability mass function, denoted by $p_X(x)$; pdf - probability density function; CDF - Cumulative distribution function, denoted by F_X.

- U is a Uniform$[0,1]$ random variable ; X is a Poisson(λ) random variable if the pmf is $\lambda^k e^{-\lambda}/k!, k = 0, 1, 2 \ldots$; X is an Gaussian(μ, σ^2) random variable if the pdf is $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}$.
- For a random variable X, the characteristic function is $\phi_X(t) := \mathbb{E}\left(e^{itX}\right), t \in \mathbb{R}$.
- A sequence of random variables X_n converges in distribution to X (denoted as $X_n \Rightarrow X$) if $F_{X_n}(x) \to F_X(x)$ at all continuity points x of F_X.