# B3 - Probability III : End-Semester Exam

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**18th November, 2025.** **Time Limit : 3 hours.** **Max. points : 40.**

**There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully. Some notations are recalled at the end.**

## 1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

**Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed. Write PART A answers in a separate page.**

**Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points. 1 point will be awarded if only some correct choices are chosen and no wrong choices are chosen.**

1. Let $\Omega = [0, \infty)$ and $\mathcal{F}$ be the $\sigma$-algebra generated by the sets $[0,1), [1,2), \ldots, [n, n+1), \ldots$. Which of the following sets belong to $\mathcal{F}$ ?

   (a) $[18, 19]$

   (b) $(18, 19)$

   (c) $[11, 12)$

   (d) $\{11, 12\}$

2. Let $X_i, i \geq 1$ be i.i.d. Rademacher random variables with mean $2p - 1$ for some $p \in [0, 1]$. Which of the following are always (i.e., for all $p$) tight sequences of random variables ?

   (a) $\sum_{i=1}^{n} X_i, n \geq 1$.

   (b) $n^{-1} \sum_{i=1}^{n} X_i, n \geq 1$.

   (c) $n^{-1/2} \sum_{i=1}^{n} X_i, n \geq 1$.

   (d) $\prod_{i=1}^{n} X_i, n \geq 1$.

3. Let $X_i, i \geq 1$ be i.i.d. random variables with mean 0, variance 1 and $\mathbb{E}(|X_1|^3) \leq 2025$. Which of the following sequences converge in distribution ?

   (a) $n^{-1} \sum_{i=1}^{n} iX_i, n \geq 1$.

   (b) $n^{-1} \sum_{i=1}^{n} X_i, n \geq 1$.

   (c) $n^{-1/2} \sum_{i=1}^{n} X_i, n \geq 1$.

   (d) $\sum_{i=1}^{n} \frac{X_i}{i}, n \geq 1$.

4. Let $X_n$ be a sequence of random variables such that $X_n \to X$ a.s.. Under which of the following additional conditions does $\mathbb{E}(X_n) \to \mathbb{E}(X)$ hold ?

   (a) $X_n \geq 0, X_{n+1} \geq X_n + \frac{1}{2^n}, \forall n \geq 1.$

   (b) $X_n \geq 0, X_{n+1} \leq X_n + \frac{1}{2^n}, \forall n \geq 1.$

   (c) $X_n \geq 0, X_{n+1} \leq X_n + \frac{1}{2^n}, \forall n \geq 1$ and $\mathbb{E}(|X_1|) < \infty$.

   (d) $|X_{n+1} - X_n| \leq \frac{1}{2^n}, \forall n \geq 1$ and $\mathbb{E}(|X_1|) < \infty$.

5. Let $\phi$ be a characteristic function. Which of the following functions are also a characteristic function ?

   (a) $\bar{\phi}$, the complex conjugate of $\phi$.

   (b) $\frac{\phi^{18}}{3} + \frac{\phi^{11}}{3} + \frac{\phi^{2025}}{3}$.

   (c) $\sqrt{\phi}$.

   (d) $-\phi$.

## 2 PART B : 30 Points.

**Answer any three questions only. All questions carry 10 points.**

**Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly.**

1. Let $X$ be the Exponential(1) random variable and $Y$ be a non-negative random variable. Show that $Y \stackrel{d}{=} X$ iff $\mathbb{E}(f'(Y)) = \mathbb{E}(f(Y)) - f(0)$ for all differentiable functions $f : [0, \infty) \to \mathbb{R}$ such that $\mathbb{E}(|f'(X)|) < \infty$.

2. (a) Show that $X$ is a symmetric random variable iff $\phi_X$ is a real-valued even function. **(5)**

   (b) Let $X, Y$ be random variables and $U$ be an independent uniform $[0, 1]$ random variable. Show that there exists a measurable function $f : \mathbb{R} \times [0, 1] \to \mathbb{R}$ such that $(X, Y) \stackrel{d}{=} (X, f(X, U))$. **(5)**

3. Fix $k \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n, \ldots$ be i.i.d. random variables such that $\mathbb{E}(|X_1|^j) < \infty$ for all $j \geq 1$ and $\mathrm{Var}(X_1) > 0$. Define $S_n := \sum_{i=1}^{n} (X_0 + \prod_{j=i}^{i+k} X_j)$.

   (a) Show that $n^{-1} S_n$ converges a.s. and find the limit. **(6)**

   (b) Is $\limsup_{n \to \infty} |S_n|/\sqrt{n} < \infty$ a.s.? **(4)**

4. Let $X_n, \geq 1$ be i.i.d. continuous random variables with CDF $F$. We say that $X_n$ is a *record value* if $X_n > X_i$ for all $i < n$. Let $I_n$ be the indicator that $X_n$ is a record value.

   (a) Show that the random variables $I_n$ are independent random variables and find their distributions. **(3)**

(b) Set $S_n = \sum_{i=1}^{n} I_i$ be the number of records upto $n$. Prove that $\frac{S_n}{\log n} \xrightarrow{a.s.} 1$.**(3)**

(c) For $S_n$ as above, show that $\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\text{Var}(S_n)}} \Rightarrow Z$, a standard normal random variable.**(4)**

5. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables with finite mean $\mu$ and finite variance $\sigma^2 > 0$. Let $N_1, \ldots, N_n$ be i.i.d. Poisson random variables with mean 1. Let $S_n := \sum_{i=1}^{N_1+\ldots+N_n} X_i$.

   (a) Show that $n^{-1}S_n$ converges a.s. and find the limit. **(3)**

   (b) Does $(S_n - n\mu)/\sqrt{n}$ converge in distribution ? If so, find the limit. **(3)**

   (c) Does $(S_n - n\mu)/\sqrt{n}$ converge in probability ? If so, find the limit. **(4)**

## Notations

- $\mathbb{N}$ - set of Natural numbers; $\mathbb{Q}$ - set of Rationals ; $\mathbb{R}$ - set of real numbers.
- pmf - probability mass function, denoted by $p_X(x)$; pdf - probability density function; CDF - Cumulative distribution function, denoted by $F_X$.
- $X$ is a Rademacher random variable if the pmf is $p_X(1) = p = 1 - p_X(-1)$ for some $p \in [0, 1]$.
- $U$ is a Uniform$[0, 1]$ random variable ; $X$ is a Poisson$(\lambda)$ random variable if the pmf is $\lambda^k e^{-\lambda}/k!, k = 0, 1, 2 \ldots$ ; $X$ is an Exponential$(1)$ random variable if the pdf is $e^{-x}, x \geq 0$.
- A random variable $X$ is symmetric if $X \stackrel{d}{=} -X$.
- A function $f : \mathbb{R} \to \mathbb{R}$ is even if $f(x) = f(-x), \forall x \in \mathbb{R}$.
- For a random variable $X$, the characteristic function is $\phi_X(t) := \mathbb{E}\left(e^{itX}\right), t \in \mathbb{R}$.
- A sequence of random variables $X_n$ converges in probability to $X$ if $\mathbb{P}(|X_n - X| \geq \epsilon) \to 0$ for all $\epsilon > 0$.
- A sequence of random variables $X_n$ is tight if for all $\epsilon > 0$, there exists a $r \geq 0$ such that $\sup_n \mathbb{P}(|X_n| \geq r) \leq \epsilon$.
- A sequence of random variables $X_n$ converges in distribution to $X$ (denoted as $X_n \Rightarrow X$) if $F_{X_n}(x) \to F_X(x)$ at all continuity points $x$ of $F_X$.

Quick recall of facts:

- An a.s. converging sequence of random variables converges in expectation if the entire sequence is dominated by an integrable random variable.
- Sum of mean zero independent random variables converge a.s. if sum of variances is finite.
- Tail events of independent random variables are trivial.
- Sum of mean zero independent random variables converges in distribution to a normal random variable, if its sum of absolute third moments vanishes asymptotically.