Algebra V midterm

Unless otherwise stated, all groups are finite and all representations finite dimensional over the field of complex numbers. Justify everything precisely.

1. Let A, B and C be modules over a commutative ring R. Prove from basic principles either the associativity or the distributivity property of tensor product of modules, i.e., one of the following: $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ and $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$.

2. Let G be an arbitrary group and V a representation of G over a field of characteristic 0. Find within the representation $V \otimes V$ two subrepresentations, respectively isomorphic to the second symmetric and exterior powers of V, and show that $V \otimes V$ is their direct sum. Is the result true over a field of any positive characteristic?

3. Consider a G-module M. To prove the following, you may use basic results from the text but you must specify precisely which result you are using and at precisely which place.

a) Suppose M is written as a direct sum of simple G-modules V_i, with multiplicities n_i. Show that the n_i are uniquely determined by M.

b) Show that M is uniquely determined by its character.

4. The dihedral group D_8 consists of the 4 rotational and 4 reflection symmetries of a square, with 5 conjugacy classes of sizes 1 (identity), 1 (rotation through 180°), 2 (other rotations), 2 (reflections through the diagonals) and 2 (other reflections).

a) Carefully construct the character table of this group by listing conjugacy classes in the order given above. Also construct all irreducible representations. Show all details. (It will be useful to know the subgroups of D_8.

b) Consider D_8 acting on the set of vertices of the square. Find the multiplicity of each simple D_8-module in the associated representation.

OR

4. a) Find the missing rows in the given character table.

b) Let the representation corresponding to χ_i be V_i. Decompose $V_2 \otimes V_3$ as a sum of irreducible G-modules.

c) How many normal subgroups does the given group have? (Do you know this group?)

Please turn over for more problems! →

6. Some optional problems of varying difficulty. You may want to do these if you couldn't do something above.

a) If for a G-module V, the only G-linear maps from V to V are scalars, then V is a simple G-module. True or false?

b) Find all one-dimensional representations of the symmetric group S_n.

c) For a group G, let M be a G-module in which each simple G-module appears with positive multiplicity. Show that M is faithful (i.e., the only element of G that acts by the identity map on M is the identity element of G). Is the converse true?

d) For a simple G-module M, consider the linear map $\mathrm{Hom}_{\mathbb{C}}(M,M) \to \mathrm{Hom}_G(M,M)$ taking any matrix f to its average under the action of G on $\mathrm{Hom}_{\mathbb{C}}(M,M)$. Which matrices are in the kernel of this map?

e) Show that over an arbitrary field, a square matrix is conjugate to its transpose.