Backpaper Examination

Physics IV - B. Math III .

Max. Marks 100 Time:3 hrs.

1. (i) In $\mathcal{H} = L^2(\mathbb{R}^3)$, along with usual position and momentum operators P_j, Q_k $(j, k = 1, 2, 3)$, define the time-reversal operator T by:

$$(Tf)(x) = \overline{f(x)} \qquad \forall f \in \mathcal{H}.$$

Show that T is an antilinear-unitary operator, i.e. $T(\alpha f + g) = \bar{\alpha} Tf + Tg, T^2 = I$, and that $(P_j T + TP_j)f = (Q_k T - TQ_k)f = 0$ for suitable $f \in \mathcal{H}$ for all j, k.

(ii) Let H be a self-adjoint Hamiltonian operator in $\mathcal{H}$ and $U_t = exp \quad (-iHt), t \in \mathbb{R}$ its associated unitary evolution. If H is of the canonical form $H = \frac{1}{2}\sum_j P_j^2 + V(\underline{Q})$, for some real-valued function V on $\mathbb{R}^3$, show that

$$TU_t = U_t^* T \quad \forall t \in \mathbb{R}.$$

(iii) Let the angular momentum operators L_j $(j = 1, 2, 3)$ be defined by the cyclic permutations: $L_1 = Q_2P_3 - Q_3P_2$, $L_2 = Q_3P_1 - Q_1P_3$ and $L_3 = Q_1P_2 - Q_2P_1$. Show that $TL_j = -L_jT$ for each j.

2. Let A be a linear operator in a separable infinite-dimensional Hilbert space $\mathcal{H}$ (with orthonormal basis $\{e_j\}_{j=1}^{\infty}$) given by

$$Ae_j = j^{-1/4} e_j \quad 1 \leq j \leq \infty$$

and by linearly extending it.

(i) Show that A is a bounded operator and estimate its norm.

(ii) Show that A is a compact operator, i.e. it maps the unit ball $B_1 = \{x \in \mathcal{H} : \|x\| \leq 1\}$ into a set whose closure is compact.

(Hint: Show that the image of the unit ball has the Bolzano-Weirstrass property).

3. In 3-dimensional Euclidean space, define the angular momentum operators $L_1 = Q_2P_3 - P_3Q_2, L_2 = Q_3P_1 - P_1Q_3,\ L_3 = Q_1P_2 - Q_2P_1$ where $\{Q_j\}_{j=1}^3$ and $\{P_j\}_{j=1}^3$ are the position and momentum operators in the Hilbert space $L^2(\mathbb{R}^3)$.

(i) Establish the commutation relations amongst L_1, L_2 and L_3.

(ii) Show that $\underline{L}^2 \equiv \sum_{j=1}^3 L_j^2$ commutes with $\{L_j\}_{j=1}^3$.

(iii) Define $L_\pm \ \equiv \ L_1 \pm iL_2$. If $\psi_{\ell,m} \in \mathcal{H}$, (where $\ell \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, \}$ and $-\ell \leq m \leq \ell$), are unit vectors such that

$$\underline{L}^2 \psi_{\ell,m} = \ell(\ell+1)\psi_{\ell,m}$$

and

$$L_3\psi_{\ell,m} = m\psi_{\ell,m},$$

show that $\quad L_\pm\psi_{\ell,m} = C_\pm(\ell,m)\psi_{\ell,(m\pm1)}$ for $-\ell \leq m \leq \ell$, and determine the constants $C_\pm(\ell,m)$.