Physics IV
ISI B.Math
Final Exam : April 29,2019

Total Marks: 70
Time : 3 hours
Answer all questions

1. (**Marks : 6 + 4 = 10**)

Consider the reaction $A \to B + C$ where a particle A of mass m_A decays into particles B and C of masses m_B and m_C respectively .

(a) If A is at rest in the lab frame, find the energy E_B of the particle B in terms of the masses of the particles A, B, C.

(b) An atom of mass M at rest decays to a state of rest energy $(M - \delta)c^2$ by emitting a photon of energy $h\nu$. Show that $h\nu < \delta$.

2. (**Marks : 8**)

A and B both start off at the origin and simultaneously head off in opposite directions at speed $\frac{3c}{5}$ with respect to the ground.
A moves to the right and B moves to the left. Consider a mark on the ground at $x = L$. As viewed from the ground frame, A and B are a distance $2L$ apart when A passes this mark. As viewed by A, how far away is B when A coincides with the mark ?

3. (**Marks : 6 + 4 + 4 = 14**)

A particle mass m moves under the influence of the potential $V(x) = 0$ if $0 \leq x \leq a$ and ∞ otherwise.

(a) Solve the time independent Schrödinger equation for this potential and find the stationary states $\psi_n(x)$ and their corresponding energies E_n.

(b) The particle starts out in the left half of the infinite square well described by the above potential and at $(t = 0)$ is equally likely to be found in any point in that region. What is the initial wave function $\Psi(x, 0)$ (assume it is real)?

(c) What is the probability that a measurement of the energy will yield the value $\frac{\pi^2\hbar^2}{2ma^2}$ with the initial wavefunction as in part (b) ?

4. (**Marks: 10**)

For a certain system, the operator corresponding to the physical quantity A does not commute with the Hamiltonian. It has eigenvalues α_1 and α_2 corresponding to properly normalized eigenfunctions

$$\phi_1 = \frac{(u_1 + u_2)}{\sqrt{2}}$$

$$\phi_2 = \frac{(u_1 - u_2)}{\sqrt{2}}$$

where u_1 and u_2 are properly normalized eigenfunctions of the Hamiltonian with eigenvalues E_1 and E_2. If the system is in the state $\psi = \phi_1$ at time $t = 0$, show that the expectation value of A at time t is

$$< A >= \left(\frac{\alpha_1 + \alpha_2}{2}\right) + \left(\frac{\alpha_1 - \alpha_2}{2}\right) \cos\left(\frac{|E_1 - E_2| t}{\hbar}\right)$$

5. (**Marks : 4 + 3 + 5 + 4 = 16**)

The annihilation operator for the harmonic oscillator in 1-d is defined as $\hat{a} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega_0}\right)$ where m is the mass and ω_0 is the angular frequency of the harmonic oscillator

(a) If $\hat{H}$ is the Hamiltonian for the above one dimensional harmonic oscillator and $\hat{N} = \hat{a}^\dagger \hat{a}$ is the number operator, show that $\hat{N}$ and $\hat{H}$ have simultaneous eigenstates. Find the eigenvalues of the energy in terms of the eigenvalues n of the number operator where $n = 0, 1, 2 \cdots$. It is not necessary to derive the eigenvalues of the number operator

(b) Given that $|n>$ is a normalized eigenstate corresponding to the eigenvalue n of the Hamiltonian of the harmonic oscillator where n is an integer and $\hat{a}|n >= \sqrt{n}|n-1>$ and $\hat{a}^\dagger|n >= \sqrt{n+1}|n+1>$, Find $<\hat{x^3}>$ for the nth energy eigenstate of the harmonic oscillator.

(c) Show that the average kinetic energy $< T >$ is equal to the average potential energy $< V >$ for any energy eigenstate of the harmonic oscillator

(d) In the case of the 1-d harmonic oscillator the energy levels are non-degenerate, i.e, an energy eigenvalue is associated with a unique eigenstate. Now if we generalize to three dimensions and consider the potentials i) $V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$ and (ii)$\frac{1}{2}m({\omega_1}^2 x^2 + {\omega_2}^2 y^2 + {\omega_3}^2 z^2)$ of the three dimensional isotropic and anisotropic harmonic oscillators, discuss whether the feature of nondegeneracy persists. You do not have to solve the entire problem in three dimensions all over again, but just have to give a plausible argument.

6. (**Marks : 3 + 3 + 3 + 3 = 12**)

Consider the angular momentum operators $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, $L_z = xp_y - yp_z$.

(a) Compute $[L_x, L_y]$,$[L_y, L_z]$,$[L_z, L_x]$

(b) Show that it is possible to measure $L^2 = L_x^2 + L_y^2 + L_z^2$ and any one component of the angular momentum simultaneously with arbitrary accuracy.

(c) Evaluate the commutators $[L_z, r^2]$ and $[L_z, p^2]$ where $r^2 = x^2 + y^2 + z^2$ and $p^2 = p_x^2 + p_y^2 + p_z^2$

(d) Show that the Hamiltonian $H = \frac{p^2}{2m} + V$ commutes with all three components of $\mathbf{L}$, provided that V depends only on r