ISI – Bangalore Center – B Math - Physics IV – End Semestral Exam
Date: 2 May 2018. Duration of Exam: 3 hours
Total marks: 100
Answer all questions.

Q1 [Total Marks: 5x4=20]

These are short questions. No detailed calculations are needed. But indicative arguments need to be given.

a.) The front and back doors of a garage are in the *y-z* planes 10 meters apart as measured in its rest frame. A ladder with 10 meters rest length is moving along the *x* axis towards the right (with respect to the garage frame). Initially both ends of the ladder are far to the left of the garage.

Is it possible for both ends of the ladder to be completely within the garage at the same time as seen in the garage frame of reference?

Is it possible for both ends of the ladder to be completely within the garage at the same time as seen in the ladder frame of reference?

b.) A woman is at rest in a frame of reference. As per her watch she claps her hands at 8 am and then flashes a beam of light 30 seconds later. An observer moving past her at great speed sees these two things happening at different places and claims that the woman clapped her hands and flashed the light at the same time. Discuss if that is possible or not.

c.) Suppose we have a quantum mechanical system with the following Hamiltonian $H = (a_+ a_- - \frac{3}{2})\hbar\omega$ where the a_+, a_- are Hermitian conjugates of each other. Assuming that eigenstates of H are normalizable, which of the following is a lower bound for eigenvalues of H $-\frac{3}{2}\hbar\omega, 0, +\frac{3}{2}\hbar\omega$? Justify your answer.

d.) Under what conditions can we find simultaneous eigenstates of momentum and energy for a single particle quantum system?

e.) Determine with appropriate reasoning if the following statement is true or false: The eigenvectors of L^2 and L_z for a three dimensional one particle quantum system will be given by $Y_l^m(\theta,\phi)$ even for a non central potential.

Q2. [Total Marks: 4+4+4+4+4= 20]

a.)A ray of light is travelling in the *xy* plane making an angle θ with the +*x* axis. Show that in another frame moving with velocity *v* in the +*x* direction with respect to the previous frame, the ray of light is moving in a direction θ' given by

$$\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta}, \ \beta = v/c$$

b.) Let U^μ be the velocity four vector defined by $U^\mu = \frac{dx^\mu}{ds}$, where *ds* is the invariant interval. Show that U^μ is a unit time like vector.

c.) The acceleration four vector is defined by $a^\mu = \frac{dU^\mu}{ds}$. Show that U^μ and a^μ are orthogonal to each other. Remember that the dot product of U^μ and a^μ is $U^0a^0 - U^1a^1 - U^2a^2 - U^3a^3$.

d.) Using the above or by direct calculation starting from the standard Lorentz transformation with boost +*v* in the x direction, show that $a'_x = \frac{a_x}{\gamma^3(1-u_x v/c^2)^3}$.

e.) Show that in general if the standard acceleration $\vec{a} = \frac{d^2\vec{x}}{dt^2}$ is zero in one frame then it is zero in all frames related by a Lorentz transformation .

Q3. [Total Marks: 2+2+2+2+4+4+4=20]

In a one dimensional quantum mechanical problem, suppose that the potential is given by

$V = -V_0, -a \le x \le a$,

$V = 0, -b < x < -a, a < x < b$ and

$V = \infty$ elsewhere. Here $a, b > 0, V_0 > 0$.

Consider an energy eigenstate with energy between $-V_0$ and 0.

a.) Write the most general form of the wave function in the region $x>a$.

b.) Write the most general form of the wave function in the region $-a<x<a$.

c.) What are the boundary conditions to be satisfied by the wave function and its first derivative at $x=a$? (Do not need to derive these boundary condition)

d.) Explain why the energy eigenstate can be chosen to be even or odd function of *x*.

e.) Assuming the existence of an energy eigenstate which is an even function of *x*, show that

$$\tanh[\kappa(b-a)] = \frac{\kappa}{k\tan(ka)} \text{ where } \kappa = \sqrt{\frac{2m|E|}{\hbar^2}} \text{ and } \kappa = \sqrt{\frac{2m(V_0-|E|)}{\hbar^2}}$$

f.) What are energy levels in part e.) as $b \to a$?
g.) Which energy levels of an infinite well match the energy levels found in part f.). Explain the reason for the match.

Q4.[Total Marks: 4+4+10+2= 20]

A quantum mechanical harmonic oscillator in one dimension is described by the Hamiltonian: $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$. Consider the operator $a_- = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + ip)$ and its Hermitian conjugate a_+.

a.) Express *H* in terms of a_+a_- and show that *H* and a_+a_- have simultaneous eigenstates.

b.) Show that if ψ_E is an energy eigentstate of H with energy *E*, then $a_-\psi_E$ is also an energy eigenstate with energy $E-\hbar\omega$.

c.)Show that there must be a state ψ_0, such that $a_-\psi_0 = 0$. Determine ψ_0 upto a normalization constant by solving $a_-\psi_0 = 0$. Indicate how to create all the other energy eigenstates from this ground state and determine their energies.

d.) Qualitatively sketch the wave function for the two lowest energy states.

Q5. [Total Marks: 3+5+2+4+2+4=20]

Consider a three dimensional single particle quantum system.

a.) Express the angular momentum operator L_z in standard spherical coordinates.

b.) Show that the eigenvalues of L_z are quantized in units of $m\hbar$ where *m* is an integer. Find the corresponding eigenvalues normalized properly.

Assume that the potential in this problem is central and that sytem is in a state described by the following wave function. $\psi(x,y,z) = C\left(1+\frac{x}{r}\right)f(r)$ where $r = \sqrt{(x^2+y^2+z^2)}$,

$\int_0^\infty r^2 |f(r)|^2 \, dr = 1$ and *C* is a real normalization constant. Use this wave function for the rest of the question .

c.) Show that the probability of the measured value of L_z to be $-\hbar$ is $\frac{2\pi}{3}C^2$.

d.) What other values of L_z will be found and what are their associated probabilities?

e.) Find the value of C.

f.) Show that the expectation value of L^2 is $\frac{1}{2}\hbar^2$.

-----------------------Some useful results----------------

1. The infinite square well of width L (V = infinite for $0<x<L$) wave function has energy levels given by $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

2. The spherical harmonics are orthonomal functions. The first few are given below.

$Y_0^0 = \sqrt{\frac{1}{4\pi}}$,

$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta, Y_1^{\pm 1} = \mp\sqrt{\frac{3}{4\pi}}(\sin\theta)e^{\pm i\phi}$

3. The spherical coordinates are given by

$x = r\sin\theta\cos\phi,$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$