Final Examination on May 6,2009
Quantum Mechanics/PhysicsIV
Time:3 hours
Indian Statistical Institute.

Answer all questions.

1. Consider a quantum Harmonic oscillator with the self-adjoint Hamilton operator in $H \equiv L^2(\mathbb{R}) : H = \frac{P^2}{2} + \frac{\omega^2}{2} Q^2$, where P and Q are the momentum and position operators in H, and ω is the frequency of the oscillator. Further more, note that $\psi_o(x) \equiv (\pi)^{-\frac{1}{4}} e^{-\frac{x^2}{2}}$ is the eigenvector corresponding to the smallest eigenvalue of H and nth eigenvector $\psi_n = (n!)^{-\frac{1}{2}} (a^*)^n \psi_o$, where $a = (2)^{-\frac{1}{2}} (\sqrt{\omega} Q + \frac{P}{\sqrt{\omega}})$.

(i) Find the expectation values $E_n(Q), E_n(P)$ of the position and momentum operators in the state ψ_n as well as their variances $\delta_n(Q) = \langle \psi_n, [Q - E_n(Q)]^2 \psi_n \rangle$ etc.

(ii) Show that $\delta_n(P)\delta_n(Q) \geq \frac{1}{4}$ for all n. (Caution:Take into account the fact that P and Q are not defined everywhere in $L^2(\mathbb{R})$, and note that $\| \psi_n \| = I$ for n=0,1,2,...).

2. With the setup of the problem 1, for $f \epsilon H$ and $z \epsilon C$, set $\hat{f}(z) = \sum_{n=0}^{\infty} \langle \psi_n, f \rangle \frac{z^n}{\sqrt{n!}}$.

(i)Show that $\hat{f}$ is an entire function.

(ii) Let $\hat{H}$, be the Hilbert space of entire functions of such that $\int_C |g(z)|^2 e^{\frac{-|z|^2}{2}} \mu(dz) < \infty$, where μ is the Lebesgue measure on the complex plane C, and let the norm be defined by

$$\| g \|^2 = (\pi)^{-1} \int_C |g(z)|^2 e^{\frac{|z|^2}{2}} \mu(dz)$$

Show that $\left\{ e_n = \frac{z^n}{\sqrt{n!}} \right\}_{n=0}^{\infty}$ forms a O.N.B. of $\hat{H}$, and

(iii) that the map $U : f \to \hat{f}$ is an isometric isomorphism between H and $\hat{H}$.

(iv) Prove that $U\psi_n = e_n$,

$$(Uaf)(z) = \frac{d}{dz}(Uf)(z),$$

and

$$(Ua^* f)(z) = z(Uf)(z)$$

for all f in the domain of a and a^*.

3. Consider the $j = \frac{1}{2}$ irreducible representation of the angular momentum algebra or of the group of rotations in three dimensions, given in terms of the three Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ satisfying relations: $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\sigma_{jk}, (f, k = 1, 2, 3)$ and $\sigma_j\sigma_k = i\sigma_\ell$, where $(jk\ell)$ is a cyclic permutation of (1,2,3).

(i) Prove the Euler's formula in $M_2(C)$ $exp\ (i\theta\sigma_u) = \cos(\theta \parallel u \parallel) + \frac{i\sigma_u}{\|u\|} \sin(\theta \parallel u \parallel)$, where $0 \leq \theta \leq 2\pi, \sigma_u = \sum_{j=1}^{3} \sigma_j u_j$ with $u\epsilon\mathbb{R}^3 \diagdown \{o\}$.

(ii) Show that $\sigma_j (j = 1, 2, 3)$ transform like the 3 coordinate axes under rotation: e.g.

$$e^{i\sigma_3 \frac{\theta}{2}} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} e^{i\sigma_3 \frac{\theta}{2}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}.$$