End Semester Examination

Physics III,
B. Math., 3^{rd} year,
September - December 2020.
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December 14^{th}, 2020, Morning Session.
Duration: 3 hours.
Total points: 75.

Please give arguments where necessary. If it is unclear from your answer why a particular step is being taken, full credit will not be awarded. Grades will be awarded not only based on what final answer you get, but also on the intermediate steps.

1. (a) Show that the electric field produced by a uniform charge density ρ confined inside a volume $\mathbb{V}$ enclosed by a surface $\partial\mathbb{V}$ can be written as

$$\vec{E}(\vec{r}) = \frac{\rho}{4\pi\epsilon_0} \oint_{\partial\mathbb{V}} \frac{dS' \hat{n}(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

where $\vec{r'} \in \partial\mathbb{V}$. Do not assume that $\mathbb{V}$ is of a particular shape. [4]

(b) Show that the electric field due to any arbitrary localized charge distribution $\rho(\vec{r'})$ can be written as

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi\epsilon_0} \int_{\mathbb{V}} d\mathbb{V}' \frac{\vec{\nabla}_{\vec{r'}} \left(\rho(\vec{r'})\right)}{|\vec{r} - \vec{r'}|}.$$

$\mathbb{V}$ is the volume over which $\rho(\vec{r'})$ is non-zero. The term *localized* should be interpreted as meaning that a surface $\partial\mathbb{V}$ can always be found completely enclosing $\mathbb{V}$ such that there is no charge on $\partial\mathbb{V}$. [4]

(c) There exists a volume $\mathbb{V}$ enclosed by the surface $\partial\mathbb{V}$. It is known that the potential takes the constant value Φ_0 all over $\partial\mathbb{V}$. It is also known that the total charge inside $\mathbb{V}$ is Q. There is no charge anywhere else. Show that the total electrostatic energy contained outside $\mathbb{V}$ is

$$U_{\text{out}} = \frac{1}{2} Q\Phi_0.$$

[4]

2. (a) Imagine that there is a finite volume in space $\mathbb{V}$ in which there is no charge anywhere. Show that the electrostatic potential $\Phi(\vec{r}), \vec{r} \in \mathbb{V}$ cannot have a local maximum or minimum. [2]

   (b) The Cartesian components of an electric field in a region of space free of charge are given by

$$E_k(\vec{r}) = C_k + D_{jk} r_j$$

   where $j, k = \{x, y, z\}$, the summation convention is followed for indices i.e., repeated indices are summed over, and r_j is the j^{th} Cartesian component of $\vec{r}$. C_k and D_{jk} are constants.

      i. Show that D_{jk} are symmetric ($D_{jk} = D_{kj} \forall j, k$) and traceless ($\sum_k D_{kk} = 0$). [4]

      ii. What is the most general form of the potential function that will generate this electric field? [2]

   (c) A solid conductor has an arbitrarily-shaped volume of cavity scooped out from its interior such that the cavity is now vacuum. Show that $\vec{E} = 0$ inside the cavity. [3]

   (d) A spherical conducting shell of radius b contains a solid conducting sphere of radius a ($b > a$). Find the capacitance (defined as $C = Q/|\triangle\Phi|$, $\triangle\Phi$ being the potential difference between the two conducting surfaces) of the configuration in the following two cases:

      i. The sphere of radius a is grounded and the shell of radius b carries a charge Q. [3]

      ii. The sphere of radius b is grounded and solid sphere of radius a carries a total charge Q. [3]

   (e) There are two infinite conducting planes located parallel to the $x - y$ plane at $z = d$ and $z = -d$. Both planes are grounded. An infinite plane with uniform surface charge density σ is interposed between them and parallel to them at an arbitrary position z, not necessarily equally distant from the grounded planes.

      i. Find the charges induced on the two planes by the plane in between them. [3]

      ii. Find the electric potential at the position of the interposed sheet of charge. [3]

      iii. Find the force per unit area which acts on the sheet of charge. [2]

3. (a) A cube filled with dielectric matter is uniformly polarized with a magnitude P and the direction of the polarization being parallel to one of its faces. Find the electric field at the centre of the cube. Compare this field to the electric field at the centre of a uniformly polarized dielectric sphere filled with the same dielectric matter and having the same magnitude of polarization. [4]

(b) A spherical conductor of total charge Q and radius R sits centred at the origin. The space outside the sphere above $z = 0$ plane has dielectric constant ϵ_1. The space outside the sphere below the $z = 0$ plane has the dieelectric constant ϵ_2. Assume linear dielectrics for both regions in space.

i. Find the electric potential everywhere. [4]

ii. Find the distribution of free charge and bound charge, wherever they are. [4]

4. (a) A current distribution produces the vector potential

$$\vec{A}(r, \theta, \phi) = \frac{\mu_0}{4\pi} \frac{A_0 \sin(\theta)}{r} \exp(-\lambda r) \hat{e}_\phi$$

What is the magnetic moment associated with this current distribution? [4]

(b) Show that a current density with a vector potential $\vec{A}(\vec{r}) = f(r)\vec{r}$ has no dipole moment. *Hint:* You may find the following identity involving $f(r)$ helpful:

$$-\vec{\nabla} \times \left(\vec{r} \times \vec{\nabla} f(r)\right) = -\vec{r}\left[\nabla^2 f(r)\right] + \vec{\nabla} f(r) + \vec{\nabla}\left[\frac{\partial f(r)}{\partial r}\right]$$

[6]

(c) A circular disc of radius R and uniform surface charge density σ spins with a constant angular speed ω. Find the relationship between the magnitudes of the dipole moment of the disc in the following two cases:

i. when the disc is rotating about an axis passing through its center and normal to its surface.

ii. when the disc is rotating about any diameter of the disc as an axis.

[4]

5. Consider a collection of N charged point particles whose positions $\vec{r_k}, k = 1 \ldots N$ are fixed in space. However the magnitude of their charges change with time, i.e., $q_k = q_k(t)$. The rate of change of the magnitude of each charge, $\dot{q}_k(t)$, is also known.

(a) Construct a simple current density that will satisfy the continuity equation. [3]

(b) Find $\vec{B}(\vec{r}, t)$ such that the electric and magnetic fields together satisfy all four Maxwell's equations in their differential form. [6]

(c) Provide a physical interpretation of the current density constructed in part (a)? [3]