Indian Statistical Institute, Bangalore
B. Math. (hons.) Second Year, Second Semester
Ordinary Differential Equations

End Term Examination — Date : 24 April 2023
Maximum marks: 100 — Time: 3 hours

Section A

Answer all the question, each question carries 10 marks.

1. Solve $y' + P(x)y = Q(x)y^n$, for $n = 0, 1, 2, \cdots$.
2. Using variation of parameters method find a particular solution of the equation $y'' + y = f(x)$.
3. Solve $y' = (1 - x^2)^{-\frac{1}{2}}$ and use it to prove $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\frac{1}{3\times 2^3} + \frac{1\times 3}{2\times 4}\frac{1}{5\times 2^5} + \ldots$.
4. Solve $2x^2y'' + x(2x+1)y' - y = 0$ by Frobenius method.

Section B

Answer any four questions from this section, each question carries 15 marks.

5. Let $p(x) \in C^1(\Omega)$ and $q_1(x), q_2(x) \in C(\Omega)$, where $\Omega = [a, b] \subset \mathbb{R}$. Further, assume $p(x) > 0$ and $q_2(x) > q_1(x)$ on Ω. Let y_1 and y_2 be the real valued solutions of differential equations $\frac{d}{dx}[p(x)\frac{dy}{dx}] + q_1(x)y = 0$ and $\frac{d}{dx}[p(x)\frac{dy}{dx}] + q_2(x)y = 0$, resepectively. Further, if x_1 and x_2 are consecutive zeros of y_1 in Ω, then prove that y_2 has at least one zero in (x_1, x_2).
6. Let us consider the following differential operator

$$\mathbf{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x),$$

on the domain $\Omega = [a, b] \subset \mathbb{R}$, where $p \in C^1(\Omega), p > 0$ and $q \in C(\Omega)$. Consider the following boundary value problems

$$\mathbf{L}y = r, \quad \text{with conditions} \quad y(a) = \alpha,\ y(b) = \beta, \qquad (0.1)$$

where α, β are real, $r \in C(\Omega)$. Consider the homogenous counterpart of problem (0.1)

$$\mathbf{L}y = 0, \quad \text{with conditions} \quad y(a) = 0,\ y(b) = 0. \qquad (0.2)$$

Then show that following alternatives hold:

(a) If (0.2) has only trivial solutions, then, there exists a unique solution of (0.1).

(b) If (0.2) has a non trivial solution, then, (0.1) has infinitely many solutions, provided that it has a solution.

7. Consider the following nonlinear differential system

$$\begin{cases} \frac{dx}{dt} = 8x - y^2, \\ \frac{dy}{dt} = -6y + 6x^2. \end{cases} \tag{0.3}$$

Show that $(0,0)$ and $(2,4)$ are critical points of the system (0.3). Further, determine the type and stability of the critical point $(2,4)$.

8. Consider the following differential system

$$\frac{dx}{dt} = ax(x^2 + y^2) - xy^3, \qquad \frac{dy}{dt} = ay(x^2 + y^2) + x^2y^2.$$

Using the Lyapunov method, check the stability of the zero solution when $a = 0, a > 0$ and $a < 0$.

9. State the Poincaré-Bendixson Theorem. Further, consider the following differential system

$$\frac{dx}{dt} = y + x\frac{f(r)}{r}, \qquad \frac{dy}{dt} = -x + y\frac{f(r)}{r}$$

where $r = x^2 + y^2$. Let $r_0 < r_1 < \ldots$ be the zeros of $f(r)$. Then, show that the above system has limit cycles corresponding to these zeros of $f(r)$.

10. Consider the problem

$$\frac{dy}{dx} = f(x,y), \qquad \text{with initial condition } \; y(x_0) = y_0, \tag{0.4}$$

where $x \in [a,b] \in \mathbb{R}$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an analytic function. Given $\theta \in [0,1]$, find the order of the following numerical scheme for the solution to the problem (0.4)

$$y_{k+1} = y_k + h[\theta f(x_k, y_k) + (1-\theta) f(x_{k+1}, y_{k+1})], \tag{0.5}$$

where h is the step size. Further, show that for $\theta \neq \frac{1}{2}$, the numerical scheme (0.5) is convergent.