Final - Optimization (2023-24)
Time: 3 hours.
Attempt all questions. The total marks is 50.

1. Let $f : \mathbf{R}^n \to \mathbf{R}$ be convex and differentiable. Show that $\mathbf{x}^*$ solves

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})$$

if and only if $\nabla f(\mathbf{x}^*) = \mathbf{0}$. **[6 marks]**

2. Consider the constrained optimization problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x}) \qquad \text{subject to } g_m(\mathbf{x}) \le 0, \quad m = 1, 2, \cdots, M, \tag{0.1}$$

where $f, g_1, \cdots, g_m : \mathbf{R}^n \to \mathbf{R}$. Corresponding to the above problem, for each choice of real numbers $\lambda_1, \cdots, \lambda_M$, is an unconstrained problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} \left[f(\mathbf{x}) + \sum_{m=1}^{M} \lambda_m g_m(\mathbf{x}). \right]$$

   (a) Write down the dual problem. **[2 marks]**

   (b) Let $\boldsymbol{\lambda}^*$ be the solution of the dual problem, and $\mathbf{x}^*$ be the solution of the primal problem (0.1). Show that if strong duality holds, then $\boldsymbol{\lambda}^*$ is exactly what is needed to make $\mathbf{x}^*$ the solution to the unconstrained problem. **[6 marks]**

3. Let $f : \mathbf{R}^n \to \mathbf{R}$. Recall that a *subgradient* of f at $\mathbf{x}$ is a vector $\mathbf{g}$ such that

$$f(\mathbf{y}) \ge f(\mathbf{x}) + < \mathbf{y} - \mathbf{x}, \mathbf{g} >, \qquad \text{for all } \mathbf{y} \in \mathbf{R}^n.$$

   (a) Show that if f is convex then there is at least one subgradient of f. **[7 marks]**

   (b) Show that a subgradient may not exist if f is not convex. **[3 marks]**

4. Consider a convex three times continuously differentiable function $f : \mathbf{R} \to \mathbf{R}$. Let ξ be the global minimizer of f. Let $\delta > 0$ and $I_\delta = [\xi - \delta, \xi + \delta]$. Assume further that $f'''(\xi) \neq 0$, and there exists $A > 0$ such that

$$\frac{|f'''(x)|}{|f''(y)|} \le A \quad \text{for all } x, y \in I_\delta.$$

Consider the sequence

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)},$$

with $|x_0 - \xi| \le \min(\delta, \frac{1}{A})$. Show that

$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^2} = \mu,$$

for some $\mu \in (0, \frac{A}{2}]$. **[8 marks]**

5. Consider the standard form polyhedron $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \ge \mathbf{0}\}$, where $\mathbf{b}$ is an $m \times 1$ vector, and assume that the m rows of the matrix $\mathbf{A}_{m\times n}$ are linearly independent.

   (a) Suppose that two different bases lead to the same basic solution. Show that the basic solution is degenerate. **[3 marks]**

   (b) Consider a degenerate basic solution. Is it true that it corresponds to two or more distinct bases? Prove or give a counterexample. **[3 marks]**

6. While solving a standard form problem, we arrive at the following tableau, with x_3, x_4, and x_5 being the basic variables:

	x_1	x_2	x_3	x_4	x_5
-10	δ	-2	0	0	0
4	-1	η	1	0	0
1	α	-4	0	1	0
β	γ	3	0	0	1

The entries α, β, γ, δ, η in the tableau are unknown parameters. For each one of the following statements, find parameter values that will make the statement true.

(a) The current solution is optimal. **[4 marks]**

(b) The current solution is feasible but not optimal. **[3 marks]**

7. The convex hull conv(X) of a set $X \subset \mathbf{R}^n$ is the intersection of all convex sets containing X.

(a) Show that conv(X) is convex. **[2 marks]**

(b) Show that conv(X) need not be closed. **[3 marks]**