Midterm Exam - Numerical Computing
B. Math I

22 February, 2024

> (i) Duration of the exam is 3 hours.
>
> (ii) The maximum number of points you can score in the exam is 90 (total= 95).
>
> (iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. Write down the output of the following commands in Octave:
   (a) (2 points) `linspace`(-3, 4, 6)';
   (b) (3 points) $x(2,:)$ and $x(:,2)$ if $x = \text{reshape}(1:9,3,3)$;
   (c) (3 points) `polyval`([1 -3 2], [2 3])
   (d) (4 points) (`ones`(2,4)-`eye`(2,4))*`diag`(2:5)*(`ones`(4,2)-`eye`(4,2))
   (e) (4 points) A=[1 2;3 4];B=[1 0;0 1]; `disp`(A.*B); `disp`(A*B)

Total for Question 1: 16

2. Write down a command or a short code in Octave to achieve the following goals:
   (a) (5 points) Display real part, imaginary part and conjugate of a complex number z. Display transpose of a matrix A over complex numbers.
   (b) (4 points) Display the plot of the function $f(x) = \sin(x) + e^x$ for x between $-\pi$ and π.

(c) (5 points) Find the point(s) of local minima of the polynomial $x^3 - 3x^2 + 2x - 1$.

Total for Question 2: 14

3. (15 points) Let $f : [0,1] \to [0,1]$ be a continuously differentiable function such that $|f'(x)| < \frac{1}{2}$ for all $x \in [0,1]$. Prove (from first principles) that for any $x_0 \in [0,1]$, the sequence defined inductively as $x_{n+1} = f(x_n)$, $n = 1, 2, \ldots$, converges to a fixed point of f.

Total for Question 3: 15

4. For each of the following equations, determine an iteration function (and an interval I) so that the conditions of the contraction mapping fixed-point theorem are satisfied (assume that it is desired to find the smallest positive root):

   (a) (7 points) $x^3 - x - 1 = 0$

   (b) (8 points) $e^{-x} - \cos x = 0$.

Total for Question 4: 15

5. (10 points) In the bisection method, let M denote the length of the interval $[a_0, b_0]$. Let $(x_0, x_1, x_2, \ldots,)$ represent the successive midpoints generated by the bisection method. Show that

$$|x_{i+1} - x_i| = \frac{M}{2^{i+2}}.$$

Also show that the number of iterations, N, required to guarantee an approximation to a root to an accuracy ε is given by

$$N > -2 - \frac{\log(\varepsilon/M)}{\log 2}.$$

Total for Question 5: 10

6. (a) (10 points) The *divide and average* method, an old-time method for approximating the square root of any positive number a, can be formulated via the iterative sequence,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

   Show that this formula is based on the Newton-Raphson method, and converges for any initial value $x_0 > 0$.

   (b) (15 points) A calculator is defective: it can only add, subtract, and multiply. Let $a > 0$. Use the Newton-Raphson Method, and the defective calculator to devise an algorithm to compute $1/a$ correct to a desired precision level. (Hint: Solve $x - \frac{1}{a} = 0$)

Total for Question 6: 25