Back Paper Examination

Numerical Computing,
B. Math., 1^{st} year,
January - April 2022.

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June 10^{th}, 2022, Morning Session.
Duration: 3 hours.
Total points: 55.

Please give arguments where necessary. If it is unclear from your answer why a particular step is being taken, full credit will not be awarded. Grades will be awarded not only based on what final answer you get, but also on the intermediate steps.

1. Heron's algorithm for computing the square root of a positive real number y is given by $x \leftarrow \frac{1}{2}(x + y/x)$, where the symbol $\leftarrow$ denotes assignment in computer programming. It is a fixed point iteration method, and the iteration is given by

$$x_{n+1} = f(x_n) = \frac{1}{2}(x_n + y/x_n)$$

Let the absolute error at the n^{th} iteration be ε_n, and the relative error be defined as $\hat{\varepsilon}_n = \frac{\varepsilon_n}{x} = \frac{(x_n - x)}{x}$.

   (a) Find a relation between $\hat{\varepsilon}_{n+1}$ and $\hat{\varepsilon}_n$. [4]

   (b) If the starting point of the iteration is $x_0 = 1$, show that $\hat{\varepsilon}_1(x) = \hat{\varepsilon}_1(1/x)$. [3]

   (c) Show that for $x_0 = 1$, the maximum relative error for Heron's algorithm for approximating the positive $\sqrt{y}$ for $y \in [1/M, M]$ for a fixed number of iterations occurs at the ends of the interval i.e., at $y = 1/M$ and at $y = M$. (*Hint:* Use your results in the previous part(s) and make use of the relevant properties of the function $\phi(x) = \frac{1}{2}(1+x)^{-1}x^2$.). [8]

2. Let $A \in \mathbb{C}^{m\times m}$ be a Hermitian matrix (i.e., $A^\dagger = A$, where $A^\dagger$ is defined as $A^\dagger = \overline{A^T}$). An eigenvector x of A is defined as $x \in \mathbb{C}^m$, $x \neq \mathbf{0}$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$ (λ is the corresponding eigenvalue).

   (a) Prove that λ is real. [3]

(b) Prove that, if x and y are eigenvectors corresponding to distinct eigenvalues of A, x and y are orthogonal. [7]

3. Let $S \in \mathbb{C}^{m \times m}$ be a skew-Hermitian matrix i.e., $S^\dagger = -S$ (see above for the definition of $S^\dagger$). Show that

   (a) The eigenvalues of S are purely imaginary. [3]

   (b) Show that $I - S$ is non-singular, I being the $m \times m$ identity matrix. [5]

   (c) Show that the Cayley transform, Q of S, defined as $Q = (I - S)^{-1}(I + S)$, is unitary. [7]

4. You are asked to integrate numerically the continuous function (you can take it to be continuous for as many orders as you need) $f(x)$ between the interval $[0, 1]$. The following information is known to you: $f(0) = 1$, $f^{(1)}(0) = 2$, $f^{(2)}(0) = 5$, $f(1) = -1$ and $f^{(1)}(1) = -2$.

   (a) Construct the Lagrange interpolation functions with the given support points. [7]

   (b) Carry out Peano's error analysis up to $n = 2$. You must show the entire analysis to get full credit. [8]