Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra-II

Mid-term Examination I | Date : 24 February 2023
Maximum marks: 100 | Time: 10.00AM-1.00PM
Instructor: B V Rajarama Bhat

(1) Fix a natural number n and a real number b. Let $B = [b_{ij}]_{1\leq i,j\leq n}$ be the matrix with $b_{i,j} = b^{i+j}$. Compute the determinant of B. Compute the rank of B. Compute the characteristic polynomial of B. [15]

(2) Compute the inverse of

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix}$$

in two different ways by considering it as composed of block matrices of sizes 2×2, 2×2 or 3×3 and 1×1 on the diagonal and using the formulae for inverses of block matrices.

(3) Fix a natural number n and let $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ be a permutation. Let P be the associated permutation matrix defined by

$$P_{ij} = \begin{cases} 1 & \text{if } \ i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases}$$

Show that the inverse of P is same as the transpose of P. Compute the determinant of P. [15]

(4) State and prove Cauchy-Schwarz inequality for inner product spaces. [15]

(5) Consider $\mathbb{R}^3$ with standard inner product. Suppose $M : \mathbb{R}^3 \to \mathbb{R}$ is the linear map defined by

$$M\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2x_1 + x_2 + x_3.$$

Let K be the kernel of M. Obtain an orthonormal basis for K. [15]

(6) Let V be a finite dimensional inner product space and let W be a subspace of V. Let P be the projection onto W. Show that (i) $\|Px\| \leq \|x\|$, for all $x \in V$. (ii) $\|Px\| = \|x\|$ if and only if $x \in W$. [15]

(7) Let $q(x) = b_0 + b_1x + \cdots + b_kx^k$ be a polynomial. If A is a matrix, $q(A)$ is defined as $b_0I + b_1A + b_2A^2 + \cdots + b_kA^k$. (i) Suppose R is an upper triangular complex matrix with diagonal entries $a_1, a_2, \ldots, a_n$. Show that the diagonal entries of $q(R)$ are $q(a_1), q(a_2), \ldots, q(a_n)$. (ii) Let A be an $n\times n$ complex matrix with characteristic polynomial

$$p_A(x) = (x - a_1)(x - a_2)\cdots(x - a_n).$$

Show that the characteristic polynomial of $B := q(A)$ is given by

$$p_B(x) = (x - q(a_1))(x - q(a_2))\cdots(x - q(a_n)).$$

[15]