Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra-II

Back paper Examination | Date : 7 June 2023
Maximum marks: 100 | Time: 10.00AM-1.00PM
Instructor: B V Rajarama Bhat

Note: Consider standard inner product on $\mathbb{R}^n$ and $\mathbb{C}^n$ unless some other inner product has been specified. Further, $M_n(\mathbb{C})$ denotes the vector space of $n \times n$ complex matrices.

(1) Let n be a natural number. Let $a, b \in \mathbb{C}$ be complex numbers. Define a matrix $A = [a_{ij}]_{1 \leq i,j \leq n}$ by

$$a_{ij} = \begin{cases} a & \text{if } i = 1; \\ b & \text{if } i = j > 1; \\ 0 & \text{otherwise.} \end{cases}$$

(i) Compute the characteristic polynomial of A. (ii) Compute the minimal polynomial of A. (iii) When is A invertible? (You should be careful and precise in your answers.) [15]

(2) Let $\mathcal{P} = \{A \in M_2(\mathbb{C}) : \text{ trace}(A) = 0\}$. For $A, B \in \mathcal{P}$ define

$$\langle A, B \rangle = \text{ trace } (A^*B).$$

(i) Show that $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathcal{P}$. (ii) Obtain an ortho normal basis for $\mathcal{P}$. [15]

(3) Let $\mathcal{V}, \mathcal{W}$ be finite dimensional inner product spaces and let $T : \mathcal{V} \to \mathcal{W}$ be a linear map. Show that there exists a unique linear map $T^* : \mathcal{W} \to \mathcal{V}$ such that

$$\langle T^*x, y \rangle = \langle x, Ty \rangle, \ \forall x \in \mathcal{W}, y \in \mathcal{V},$$

where the inner products are taken in appropriate spaces. [15]

(4) Let $\mathcal{S}$ be the range of the matrix:

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 7 \end{bmatrix}.$$

Write down the projections onto $\mathcal{S}$ and $\mathcal{S}^\perp$ in the standard basis. Justify your answer. [15]

(5) Show that a matrix $A \in M_n(\mathbb{C})$ commutes with every matrix $B \in M_n(\mathbb{C})$ if and only if there exists $a \in \mathbb{C}$ such that $A = aI$. [15]

(6) State and prove singular value decomposition (SVD) theorem for square matrices. (You may use polar decomposition theorem.) [15]

(7) Obtain the Jordan decomposition for the following matrix (You should also write down a non-singular matrix which yields the decomposition):

$$C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

[15]