Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra-II

Final Examination | Date : 24 April 2023
Maximum marks: 100 | Time: 10.00AM-1.00PM
Instructor: B V Rajarama Bhat

Note: Consider standard inner product on $\mathbb{R}^n$ and $\mathbb{C}^n$ unless some other inner product has been specified.

(1) Let m, n be natural numbers with $m < n$. Let $a, b \in \mathbb{C}$ be complex numbers. Define a matrix $A = [a_{ij}]_{1 \leq i,j \leq n}$ by

$$a_{ij} = \begin{cases} a & \text{if } 1 \leq i, j \leq m; \\ b & \text{if } m < i, j \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

(i) Compute the characteristic polynomial of A. (ii) Compute the minimal polynomial of A. (iii) When is A invertible? (You should be careful and precise in your answers.) [15]

(2) Let F be a vector space of real polynomials defined by $F = \{p : p(x) = a_1x + a_2x^2 + a_3x^3, a_1, a_2, a_3 \in \mathbb{R}\}$. (i) Show that $\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)dt$ defines an inner product on F. (ii) Obtain an orthonormal basis for F. [15]

(3) Let P, Q be orthogonal projections on a finite dimensional complex Hilbert space $\mathcal{H}$. Take $\mathcal{M}$ = range (P) and $\mathcal{N}$ =range (Q). Show that if $\mathcal{M} \subseteq \mathcal{N}$ then $P \leq Q$ (in the sense that $Q - P$ is a positive operator.) [15]

(4) Let V, W be finite dimensional inner product spaces and let $A : V \to W$ be a linear map. Show that

$$\text{kernel } (A) = (\text{range } (A^*))^{\perp}.$$

[15]

(5) Write down the spectral decomposition in the form $\sum_j a_jP_j$, where P_j's are orthogonal projections, for the following matrices

$$A = \begin{bmatrix} 3 & i \\ -i & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

[15]

(6) Let N be a normal matrix. Show that a matrix M commutes with N and if only if it commutes with N^*. [15]

(7) Write down polar and singular value decompositions for the following matrix:

$$C = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix}.$$

[15]