Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra-II

Final Examination — Date : 25 May 2022
Maximum marks: 100 — Time: 10.00AM-1.00PM
Instructor: B V Rajarama Bhat

Note: Consider standard inner product on $\mathbb{R}^n$ and $\mathbb{C}^n$ unless some other inner product has been specified.

(1) Consider a block matrix

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, D are square matrices. Show that: (i) If D is invertible, then

$$\det(P) = \det(A - BD^{-1}C).\det(D).$$

(ii) If A is invertible, then

$$\det(P) = \det(A).\det(D - CA^{-1}B).$$

[15]

(2) Let A be an $n \times n$ matrix (with $n \geq 1$). Show that

$$\langle x, y \rangle_A := \langle x, Ay \rangle, \ \forall x, y \in \mathbb{C}^n$$

defines an inner product on $\mathbb{C}^n$ if and only if A is strictly positive (positive and invertible). [15]

(3) Let M be the subspace of $\mathbb{C}^3$ defined by

$$M = \text{ span } \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Explicitly write down the projection onto M (as a matrix in the standard basis). [15]

(4) Let V, W be finite dimensional inner product spaces and let $A : V \to W$ be a linear map. Show that

$$[\text{Range } (A)]^{\perp} = \ker(A^*),$$

where A^* denotes the 'Hermitian adjoint' of A. [15]

(5) Write down spectral decomposition, polar decomposition and singular value decomposition of following matrices.

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

[15]

(6) If B is a positive matrix show that

$$C = \begin{bmatrix} B & -B \\ -B & B \end{bmatrix}$$

is a positive matrix. (Hint: You may use the spectral theorem for B, though this is not necessary.) [15]

(7) Let

$$N = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 3 \end{bmatrix}.$$

(i) Compute the eigenvalues of N and find their geometric and algebraic multiplicities. (ii) Use Cayley-Hamilton theorem to compute the inverse of N. [15]