Midterm Exam - Harmonic Analysis (Elective)
B. Math III

26 February, 2024

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 110).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. (20 points) Prove that every finite abelian group is isomorphic (not canonically) to its dual group $\hat{G}$.

Total for Question 1: 20

2. With justification, for $a, b \in \mathbb{R}\backslash\{0\}$, compute the convolution $f_1 * f_2$ in $L^1(\mathbb{R}, dx)$ if:
   (a) (10 points) $f_1(x) = \frac{1}{x^2+a^2}, f_2(x) = \frac{1}{x^2+b^2}$;
   (b) (10 points) $f_1(x) = e^{-x^2/2a}, f_2(x) = e^{-x^2/2b}$.

Total for Question 2: 20

3. (10 points) A *generalized character* on a topological group G is defined to be a continuous homomorphism of of G into the multiplicative group of $\mathbb{C}$ (denoted by $\mathbb{C}^\times$). With justification, find the generalized characters and unitary characters of $\mathbb{C}^n, \mathbb{C}^\times$.

Total for Question 3: 10

4. (20 points) Prove that the Fourier transform gives a homeomorphism of the Schwartz space, $\mathcal{S}(\mathbb{R})$, of rapidly decreasing functions.

Total for Question 4: 20

5. (20 points) Prove that the family of Schwartz functions $\{e^{-(x-a)^2)}\}_{a\in\mathbb{R}}$ spans a dense subspace of $\mathcal{S}(\mathbb{R})$.

Total for Question 5: 20

6. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers such that

$$|a_n| \leq Cn^p, \text{ for some } p \in \mathbb{N}.$$

(a) (10 points) Prove that the sequences $S_N = \sum_{n=-N}^{N} a_n e^{2\pi i n x}, T_N := \sum_{n=-N}^{N} a_n \delta_n$ converge in the sense of tempered distributions.

(b) (10 points) Prove (from first principles) that

$$\sum_{n\in\mathbb{Z}} e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} \delta_n,$$

in the sense of tempered distributions.

Total for Question 6: 20