Midterm Exam - Function Spaces
B. Math III

15 September, 2023

- (i) Duration of the exam is 3 hours.
- (ii) The maximum number of points you can score in the exam is 100 (total = 110).
- (iii) You are not allowed to consult any notes or external sources for the exam.

Name: ____________________

Roll Number: ____________________

1. (20 points) Show that $A \subseteq \mathbb{R}^n$ is convex, if and only if $\alpha A + \beta A = (\alpha + \beta)A$ holds, for all $\alpha, \beta \geq 0$.

Total for Question 1: 20

2. For $t \geq 0$, let

$$A(t) := \left(\int_0^t e^{-x^2} \, dx \right)^2, B(t) := \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2} \, dx.$$

   (a) (10 points) Prove that $A(t) + B(t) = \frac{\pi}{4}$ for all $t \geq 0$.

   (b) (10 points) Prove that $e^{-x^2} \in L^1(\mathbb{R}_{\geq 0}; dx)$ and $\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$.

   (N.B.: Carefully justify each step, such as existence of integral, interchange of limits and integrals, etc.)

Total for Question 2: 20

3. (20 points) For $p > 0$, show that $\frac{x^{p-1}}{1-x} \log \frac{1}{x} \in L^1([0,1]; dx)$ and

$$\int_0^1 \frac{x^{p-1}}{1-x} \log \frac{1}{x}\, dx = \sum_{n=0}^{\infty} \frac{1}{(n+p)^2}.$$

Total for Question 3: 20

4. (20 points) Let $\{r_1, \ldots, r_n, \ldots\}$ be an enumeration of the set of rational numbers in $[0,1]$ and let $I_n := [r_n - \frac{1}{4^n}.r_n + \frac{1}{4^n}] \cap [0,1]$. Let $f(x) = 1$ if $x \in I_n$ for some n, and let $f(x) = 0$ otherwise. Show that f is an upper function whereas $-f$ is not an upper function.

Total for Question 4: 20

5. (20 points) Prove that the function,

$$\frac{1}{1 + x^2 \sin^2 x},$$

is not Lebesgue-integrable on $[1, \infty)$.

Total for Question 5: 20

6. (10 points) Show that $\log \frac{1}{1-x} \in L^1([0,1]; dx)$ and with justification, compute the following integral:

$$\int_0^1 \log \frac{1}{1-x}\, dx.$$

Total for Question 6: 10