Final Exam - Function Spaces
B. Math. III

15 November, 2023

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 110).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ____________________

Roll Number: ____________________

For $f \in L^1[-\pi, \pi]$, the Fourier coefficients of f are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

The series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the *Fourier series* of f. The nth partial sum (Cesaro mean, respectively) of the Fourier series for f is denoted by $s_n(f)$ ($\sigma_n(f)$, respectively).

1. Let $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, and let $\mathscr{A}$ be the algebra of functions on S^1 of the form,

$$f(z) = \sum_{n=0}^{N} c_n z^n,$$

where $c_n \in \mathbb{C}$.

(a) (10 points) Show that $\mathscr{A}$ separates points on S^1 and vanishes at no point of S^1.

(b) (5 points) Show that there are continuous complex-valued functions on S^1 which are not in the uniform closure of $\mathscr{A}$.

Total for Question 1: 15

2. (20 points) Assume that f is 2π-periodic and continuous on $\mathbb{R}$. Prove that the sequence of Césaro means of the Fourier series for f converges uniformly on $\mathbb{R}$ to f.

Total for Question 2: 20

3. (20 points) Suppose that f is a 2π-periodic function on $\mathbb{R}$ that satisfies the Lipschitz condition of order α ($0 < \alpha \leq 1$); that is $|f(x+h) - f(x)| \leq C|h|^\alpha$ for $C > 0$ independent of x. Show that if a_n, b_n are Fourier coefficients of f, then

$$a_n = O(n^{-\alpha}), b_n = O(n^{-\alpha}).$$

Total for Question 3: 20

4. (20 points) Let $f \in L^1([-\pi, \pi])$ and $x \in (-\pi, \pi)$ such that $f(x) \neq \pm\infty$. Then x is called a Lebesgue point for f if

$$\lim_{r\to 0} \frac{1}{r} \int_x^{x+r} |f(t) - f(x)| \, dt = 0.$$

For $x \in (-\pi, \pi)$ a Lebesgue point for f, show that

$$\lim_{N\to\infty} \sigma_N(f)(x) = f(x).$$

Total for Question 4: 20

5. (20 points) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Total for Question 5: 20

6. (15 points) If f is a 2π-periodic function in $C^1(\mathbb{R})$, then show that

$$\|f - s_N\|_\infty = o\left(\frac{1}{\sqrt{N}}\right).$$

(In other words, the error term for uniform approximation of f via s_N declines like "little oh" of $\frac{1}{\sqrt{N}}$.)

Total for Question 6: 15