Field and Galois Theory
Backpaper Exam
Dec. 2023

Time: 3 hours

Max score: 100

Answer **any 5** questions.

1. (a) Define constructible real numbers.
   (b) Show that if a is a constructible real number, then it is algebraic, and its degree over $\mathbb{Q}$ is a power of 2.
   (c) Show that it is impossible to construct a regular 9-gon with a compass and straight-edge. (Hint: Use part (b)). (2+8+10)

2. Suppose that $\mathbb{F}$ is a finite field of characteristic p.
   (a) Show that every element, $\alpha \in \mathbb{F}$ is of the form b^p for some $b \in \mathbb{F}$.
   (b) Prove that a polynomial over $\mathbb{F}$ is separable if and only if it is the product of distinct irreducible polynomials over $\mathbb{F}$. (8+12)

3. (a) Prove the existence and uniqueness of a field of order p^n for any prime p and any positive integer n.
   (b) If we denote the field in part (a) as $\mathbb{F}_{p^n}$, show that the extension $\mathbb{F}_{p^n}/\mathbb{F}$ is Galois. Find the Galois group of $\mathbb{F}_{p^n}/\mathbb{F}$. (10+10)

4. Let K/F be a field extension with $\text{char}(F) = p$, $p \neq 0$. Let α be a root in K of an irreducible polynomial $f(x) = x^p - x - a$ over F, $a \neq 0$.
   (a) Prove that $\alpha + 1$ is also a root of $f(x)$.
   (b) Show that $f(x)$ is separable over F.
   (c) Prove that the Galois group of $f(x)$ over F is cyclic of order p. (2+6+12)

5. (a) State the fundamental theorem of Galois theory.
   (b) Show that the Galois group of $x^5 - 4x - 1$ over $\mathbb{Q}$ is S_5. (8+12)

6. Prove that the Galois group of $x^7 + 7x^4 + 14x + 3$ is the alternating group A_7. (Hint: Use Dedekind's theorem on Galois groups of polynomials over $\mathbb{Q}$). (20)