Mid Semester Examination

Electrodynamics,
B. Math., 2^{nd} year,
January - April 2023.
Instructor: Prabuddha Chakraborty (prabuddha@isibang.ac.in)

February 23^{rd}, 2023, Morning Session.
Duration: 180 minutes.
Total points: 100.

Please give arguments where necessary. If it is unclear from your answer why a particular step is being taken, full credit will not be awarded. Grades will be awarded not only based on what final answer you get, but also on the intermediate steps.

1. (a) Let a volume $\mathbb{V}$ of arbitrary shape, enclosed by a surface $\mathbb{S}$, carry a uniform charge density ρ. The resultant electric field, everywhere, can be written as an integral over the surface alone! Show that the electric field everywhere can be written as

$$\vec{E}(\vec{r}) = \frac{\rho}{4\pi\epsilon_0}\int_{\mathbb{S}} \frac{d\vec{S}\left(\vec{r'}\right)}{|\vec{r}-\vec{r'}|}$$

   (b) Consider the same geometry as the previous problem but now let the charge density be an arbitrary smooth function of the position inside the enclosed **finite** space $\mathbb{V}$. Show that the electric field everywhere can be written as

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\int_{\mathbb{V}} dV' \frac{\nabla_{\vec{r'}}\rho(\vec{r'})}{|\vec{r}-\vec{r'}|}$$

   5 + 5 = 10 points

2. Suppose a charge density $\rho(\vec{r})$ be entirely contained in a finite spherical volume $\mathbb{V}$. Do not assume any symmetry of the charge distribution.

   (a) Derive the relationship between the dipole moment of the sphere and the average of the electric field over the sphere.

   (b) Is the spherical shape of $\mathbb{V}$ necessary for the result in the previous part? Explain why or why not.

(c) Consider a point dipole $\vec{p}$ at the origin. For **any** sphere that encloses the origin, the relationship you derived in the first part should be true. Argue that the far-field ($r \gg r'$) expression of the dipolar electric field (obtained by taking the gradient of $\frac{\vec{p}\cdot\vec{r}}{r^3}$) does not satisfy the result in part (a).

(d) Resolve the contradiction between the first and the third parts by supplementing (by hand) the electric field expression.

5 +2 + 3 + 5 = 15 points

3. Consider a sphere with radius R with the following charge density on its surface:

$$\sigma(r, \theta, \phi) = \sigma_0 \sin^2\theta \sin^2\phi$$

Find the potential $\varphi(r, \theta, \phi)$ everywhere. (Hint: Think about trying an expansion in the proper co-ordinate system).
15 points

4. A short piece of wire of length L carrying charge Q, is placed along the z-axis, centred at the origin. The detailed charge distribution $\lambda(z)$ of the wire is not known, except the charge density is an even function of z. A new length-scale $\mathcal{L}_0$ can be defined in the problem, which captures the rms length of the charge distribution:

$$Q\mathcal{L}_0^2 = \int_{\textbf{wire}} dz z^2 \lambda(z) \tag{1}$$

(a) Find the potential everywhere upto and including quadrupole order.

(b) If, instread, $\lambda(z)$ was odd in z, what would the potential be upto and including quadrupole order?

7 + 8 = 15 points

5. You have two non-intersecting conducting spheres, of radii R_1 and R_2 respectively, and an amount of charge Q to divide between them. In conductors, the charge will stay on the surface and each surface is, by itself, an equipotential surface (does not apriori mean both are at the same potential, though). Assume that the charge distribution on one sphere is not affected by the other. Find the charge division that minimizes the potential energy of the configuration. Explain the motion of charges if, after achieving electrostatic equilibrium, the spheres are now connected with a thin conducting wire (neglect any electrostatic effects by and within the wire).
8 points

6. A particular charge distribution causes the following radial electrostatic field:

$$\vec{E}(\vec{r}) = A\frac{\exp(-r/\lambda)}{r}\hat{e}_{\vec{r}}$$

where A and λ are constants with appropriate dimensions.

(a) Find the charge density that caused this electric field and give a rough sketch of the charge density.

(b) Find the total charge of the electrostatic configuration.

7 + 5 = 12 points

7. Two infinite sheets of charge, one with uniform charge density σ and the other with uniform charge density $-\sigma$ intersect each other at right angles. Find the electric field everywhere and make a rough sketch of the electric field lines.
10 points

8. Consider two infinite parallel wires along te z-direction with uniform charge densities λ and $-\lambda$ separated by a distance ℓ. Find the potential and the electric field at a point P(ρ, ϕ, z). Draw the equipotential lines in the limit $\rho \gg \ell$.
9 + 6 = 15 points

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}\left(3x^2-1\right)$
3	$\frac{1}{2}\left(5x^3-3x\right)$
4	$\frac{1}{8}\left(35x^4-30x^2+3\right)$
5	$\frac{1}{8}\left(63x^5-70x^3+15x\right)$
6	$\frac{1}{16}\left(231x^6-315x^4+105x^2-5\right)$
7	$\frac{1}{16}\left(429x^7-693x^5+315x^3-35x\right)$
8	$\frac{1}{128}\left(6435x^8-12012x^6+6930x^4-1260x^2+35\right)$
9	$\frac{1}{128}\left(12155x^9-25740x^7+18018x^5-4620x^3+315x\right)$
10	$\frac{1}{256}\left(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63\right)$

Analytic expressions for the first few orthonormalized Laplace spherical harmonics $Y_\ell^m : S^2 \to \mathbb{C}$ that use the Condon–Shortley phase convention:

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\,\sin\theta\,e^{-i\varphi}$$

$$Y_1^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\,\cos\theta$$

$$Y_1^1(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\,\sin\theta\,e^{i\varphi}$$

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\,e^{-2i\varphi}$$

$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\,\sin\theta\,\cos\theta\,e^{-i\varphi}$$

$$Y_2^0(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2\theta - 1)$$

$$Y_2^1(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\,\sin\theta\,\cos\theta\,e^{i\varphi}$$

$$Y_2^2(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\,e^{2i\varphi}$$